Combinatorics Meets Model Theory

Cambridge, June 20-24 2022

This workshop is supported by the University of Cambridge, the Heilbronn Institute for Mathematical Research under its HIMR Focused Research workshop scheme, and organised in partnership with the Clay Mathematics Institute.

All talks will be held in MR4 at the Centre for Mathematical Sciences except where indicated otherwise.

Monday, 20 June

9.00 – 9.10	Welcome
9.15 – 10.10	**Terence Tao**
	Tutorial: *The circle method from the perspective of higher order Fourier analysis*
10.15 – 10.40	Tea & coffee
10.40 – 11.20	**Daniel Palacín**
	Open problem focus: *Around pseudofinite groups*
11.30 – 12.25	**Ehud Hrushovski**
	A question on probability and amalgamation
12.30	Lunch
13.00 – 15.00	Free time
15.00 – 15.30	Tea & coffee
15.30 – 16.25	**Gabriel Conant**
	Tutorial: *Model-theoretic stability I*
16.35 – 17.15	**Caroline Terry**
	Higher order stability and regularity
17.45 – 18.45	City walk, departing from CMS with destination Clare College
19.00 – 21.00	Workshop dinner at Clare College
	Memorial Court, Queens Road, Cambridge CB3 9AJ

Tuesday, 21 June

9.00 – 9.55	**Boris Bukh**
	Tutorial: *Turán problem in graphs*
10.05 – 10.35	**Samuel Braunfeld**
	Structure and non-structure in hereditary classes
10.40 – 11:05	Tea & coffee
11.05 – 11.35	**Tingxiang Zou**
	The spatial orchard problem on cubic surfaces
11.45 – 12.25	**Jozsef Solymosi**
	Geometric triples in the plane
12.30	Lunch
13.00 – 15.00	Free time
15.00 – 15.30	Tea & coffee
15.30 – 16.25	**Gabriel Conant**
	Tutorial: *Model-theoretic stability II*
16.35 – 17.15	**Henry Towsner**
	Topology and hypergraph regularity

Wednesday, 22 June

9.00 – 9.55	**Matthew Tointon**,
	Tutorial: *Approximate groups - history, basic properties, and some recent applications*
10.05 – 10.35	**Arturo Rodriguez Fanlo**
	Generics, closed approximate subgroups and a product theorem
10.40 – 11:05	Tea & coffee
11.05 – 11.35	**Mauro Di Nasso**
	The use of nonstandard natural numbers as ultrafilters in Arithmetic Ramsey Theory
11.45 – 12.25	**Maryanthe Malliaris**
	Open problem focus: *On stability and instability*
12.30	Lunch
13.00 – 15.00	Free time
15.00 – 15.30	Tea & coffee
15.30 – 16.30	Free time
16.30 – 17.15	Open problem session
17.15 – 18.00	Wine reception at CMS

Thursday, 23 June

9.00 – 9.55	**Amador Martin-Pizarro**
	Tutorial: *How to find groups? The group configuration*
10.05 – 10.35	**Yifan Jing**
	Measure growth gap in compact simple Lie groups
10.40 – 11:05	Tea & coffee
11.05 – 11.35	**Tim Gowers** (MR2)
	Open problem focus: *The Erdős-Hajnal problem*
11.45 – 12.25	**Leonardo Coregliano** (MR2)
	Ramsey's Theorem in the countable and the approximate Erdős–Hajnal property
12.30	Lunch
13.00 – 15.00	Free time
15.00 – 15.30	Tea & coffee
15.30 – 16.25	**Anand Pillay** (MR2)
	Continuous stable regularity
16.35 – 17.15	Open problem session (MR2)
19.00 – 21.00	Dinner at Madingley Hall

Friday, 24 June

9.00 – 9.55	**Sasha Razborov**
	Natural quasi-randomness
10.05 – 10.35	**Itay Kaplan**
	On large externally definable sets and NIP
10.40 – 11:05	Tea & coffee
11.05 – 11.35	**Ben Green**
	Open problem focus: *Removal lemmas*
11.45 – 12.25	**Alexis Chevalier**
	An algebraic hypergraph regularity lemma
12.30	Lunch
13.00 – 15.00	Free time
15.00 – 15.30	Tea & coffee

Abstracts

Samuel Braunfeld, Charles University Prague [Tuesday 10.05 – 10.35]
Structure and non-structure in hereditary classes

A structure is monadically NIP if it remains NIP (a property closely related to bounded VC-dimension) under arbitrary colorings of its points. We will see how monadic NIP or its failure manifest in a structure itself rather than just in its colorings, how this generalizes some properties studied in structural graph theory, and how this can be used to produce structure or non-structure in hereditary classes. Analogous results for monadic stability might also be discussed. This is joint work with Chris Laskowski.

Boris Bukh, Carnegie Mellon University [Tuesday 9.00 – 9.55]
Tutorial: *Turán problem in graphs*

Let F be a fixed graph. How big can an n-vertex graph not containing F be? This question of Turán has been central in extremal combinatorics. I will give a brief survey, focusing on the geometric aspects.

Alexis Chevalier, University of Oxford [Friday 11.45 – 12.25]
An algebraic hypergraph regularity lemma

In "Expanding polynomials over finite fields..." (2012), Tao proves the algebraic regularity lemma, which is a strong form of the classical Szemerédi regularity lemma in the case of definable graphs in the language of rings in finite fields. The algebraic regularity lemma improves the Szemeédi regularity lemma by providing definable regular decompositions of the definable graphs which have no irregular pairs and such that the error bounds on regularity vanish as the size of the finite field grows.

Tao asks if the algebraic regularity lemma can be extended to definable hypergraphs. We answer this question positively by giving a new analysis of the algebraic regularity lemma. We use the model theory of pseudofinite fields to relate the combinatorial notion of regularity (for graphs and for hypergraphs) to Galois-theoretic information associated to definable sets. With this new analysis in hand, the algebraic hypergraph regularity lemma follows by classical results of Gowers, albeit with some interesting technical details.

Gabriel Conant, The Ohio State University [Monday and Tuesday 15.30 – 16.25]
Tutorial: *Model-theoretic stability*

One of the main pillars of model theory is *stability theory*, which emerged from seminal work of Morley on uncountable categoricity of first-order theories, and the subsequent classification program developed by Shelah. Despite the considerable breadth and depth of this field of study, stability starts with the relatively simple combinatorial notion of a stable formula, which can essentially be viewed as a bipartite graph admitting a finite bound on the size of an induced half-graph.

The goal of this tutorial is to introduce stability and explain its model-theoretic significance, while using connections to the fields of combinatorics and functional analysis as motivation. I will also explain how stability emergences naturally in these fields, and thus has appeared in general results in which no explicit assumption of stability is made.

Part I (Monday): The "local theory" of a stable formula. I will start with the necessary definitions, along with examples and non-examples. I will then state the *Fundamental Theorem of Stability*, which characterizes stability of a formula with respect to a complete theory by means of type-counting and definability of types. In the remaining time, I will aim to give the full proof of this theorem, using only basic combinatorics of half-graphs.

Part II (Tuesday): Topological perspectives on stability. I will define the notion of a "stable function", and elucidate a result of Grothendieck in functional analysis (pre-dating much of modern stability theory), which can be used to recover aspects of Monday's fundamental theorem. I will then describe the route from Grothendieck's result to a stability-theoretic lemma of Hrushovski, first proved (using different methods) as part of his work on the structure of approximate groups.

No technical knowledge of model theory or stability will be assumed. Previous exposure to the definitions of first-order structures and formulas may be helpful.

Leonardo Coregliano, Institute for Advanced Study [Thursday 11.45 – 12.25]
Ramsey's Theorem in the countable and the approximate Erdős–Hajnal property

The celebrated Erdős–Hajnal Conjecture says that in any proper hereditary class of finite graphs we are guaranteed to have a clique or anti-clique of size n^c, which is a much better bound than the logarithmic size that is provided by Ramsey's Theorem in general. On the other hand, in uncountable cardinalities, the model-theoretic property of stability guarantees a uniform set much larger than the bound provided by the Erdős–Rado Theorem in general. However, in the case of countable structures balanced between these two, all structures seem to behave in the same way since the infinite version of Ramsey's Theorem already yields a uniform set of the maximum possible cardinality.

By instead considering a different notion of large sets, namely, that of positive upper density and ignoring negligible errors, we show that the same phenomenon happens in the countable: a countable graph has a large almost clique or a large almost anti-clique if and only if it has a large almost stable set. Similarly, in the countable we can consider a variant Erdős–Hajnal property: we say that a hereditary class of finite graphs C has the approximate Erdős–Hajnal property (AEHP) if every countable graph whose finite induced subgraphs are all in C must contain a large almost clique or a large almost anti-clique. Surprisingly, AEHP has a simple characterization as precisely those classes that avoid some recursive blow-up of the 4-cycle.

In this talk, I will explain how these problems are reduced to problems for limits of dense graph sequences (graphons) and show how the corresponding graphon problems have very clean combinatorial proofs. No background knowledge in model theory or in the theory of graphons will be required.

This talk is based on joint work with Maryanthe Malliaris.

Mauro Di Nasso, Università Pisa [Wednesday 11.05 - 11.35]
The use of nonstandard natural numbers as ultrafilters in Arithmetic Ramsey Theory

Algebra in the space of ultrafilters has proved to be a powerful tool in Ramsey's theory. The hypernatural numbers of nonstandard analysis can play the role of ultrafilters, with the advantage of simplifying the notation and the possibility of studying certain partition regularity properties on the integers in an efficient manner. I will give explicit examples, including a very brief demonstration of the existence of monochromatic exponential triples a, b, b^a in any finite coloring of natural numbers.

Ben Green, University of Oxford [Friday 11.05 – 11.35]
Open problem focus: *Removal lemmas*

We give a short introduction to removal lemmas in graphs, hypergraphs and abelian groups, highlighting the key results and main open problems.

Tim Gowers, Collège de France and University of Cambridge [Thursday 11.05 – 11.35]
Open problem focus: *The Erdős-Hajnal problem*

The Erdős-Hajnal problem asks whether the bounds for Ramsey's theorem can be dramatically improved for graphs that do not contain an induced copy of some fixed graph H. The problem in general is still wide open, but there are some interesting partial results, some of which I shall describe.

Ehud Hrushovski, University of Oxford [Monday 11.30 – 12.25]
A question on probability and amalgamation

Amalgamation of two structures arose as a fundamental property in the model theory of the 1950's. For Robinson it was one of the criteria enabling quantifier elimination. Canonical amalgamation in stable theories, and higher amalgamation of their models, were two central components of Shelah's work on the number of isomorphism types in the 1970's and 1980's; three-amalgamation was seen, in the 1990's, to hold the key to the larger class of simple theories. In these settings, a certain notion of independence is available, and higher amalgamation is attempted for independent structures. In three-amalgamation, an amalgam is already assumed given for each edge of a triangle of independent structures; the problem is to combine this data into one. In probabilistic settings an alternative notion of independence is available, namely statistical independence. Here higher amalgamation is closely related to Szemeredi's lemma for graphs and for hypergraphs. These topics were very well-studied since the early 2000s in many guises, both in general (work of Gowers, Rodl-Skokan, Tao, Lovasz et al, Diaconis-Janson, Austin, Towsner is especially relevant); and for specific theories of classes (Tao, Chernikov-Starchenko, Malliaris-Shelah, Chevalier-Levi...) Nevertheless it seems to me that a basic model-theoretic question in the logic of probability still remains, related to 4- and higher amalgamation. I will aim to present it and some related material (without assuming knowledge of most of the words above).

Yifan Jing, University of Oxford [Thursday 10.05 – 10.35]
Measure growth gap in compact simple Lie groups

Let G be a compact simple Lie group of dimension d, μ is the normalized Haar measure on G, and $A, A^2 \subseteq G$ are measurable. We show that $\mu(A^2) \geq (2+\eta)\mu(A)$ when $\mu(A) < c$ with η and $c = c(d)$ quantitatively determined. I will also discuss the connections to approximate groups and to model theory. This is based on joint work with Chieu-Minh Tran.

Itay Kaplan, The Hebrew University of Jerusalem [Friday 10.05 – 10.35]
On large externally definable sets and NIP

In this talk I'll discuss the question of whether an uncountable externally definable set contains an infinite definable subset. This question was posed for NIP theories by Chernikov and Simon. All notions will be explained in the talk. Joint work with Martin Bays, Omer Ben-Neria and Pierre Simon.

Maryanthe Malliaris, University of Chicago [Wednesday 11.45 – 12.25]
Open problem focus: *On stability and instability*

This talk will be about several different views of "instability" (and time permitting, other dividing lines).

Amador Martin-Pizarro, Albert-Ludwigs-Universität Freiburg [Thursday 9.00 – 9.55]
Tutorial: *How to find groups? The group configuration*

Elekes and Szabó showed that a 3-dimensional polynomial correspondence over the complex numbers which projects generically onto any two coordinates with finite fibers such that it contain "too many" points must indeed arise generically (up to rational maps) as the group law of a 1-dimensional group, which must be abelian, as noticed by Breuillard and Wang.

At the core of the proof of Elekes-Szabó's theorem lies Hrushovski's (abelian) group configuration theorem, which is one of the key tools in geometric stability theory in order to understand and produce groups definable in an ambient stable theory. We aim to give an idea of the proof of Hrushovski's group configuration theorem, using Elekes-Szabó's theorem as a guideline to introduce some relevant notions such as modularity.

Daniel Palacín, Universidad Complutense de Madrid [Monday 10.40 – 11.20]
Open problem focus: *Around pseudofinite groups*

A pseudofinite group is an infinite group which satisfies every first-order sentence in the language of groups that is true of all finite groups. Likewise, one can define the concept of pseudofinite field, ring, graph, etc. Roughly speaking, pseudofinite structures can be seen as logical limits of finite structures. This permits to establish some transfer principle between finite and infinite objects. In this talk I will focus on pseudofinite groups and give an overview of basic background around them, and present some (non-)examples as well. Afterwards, I will state some open problems in the topic.

Anand Pillay, University of Notre Dame [Thursday 15.30 – 16.25]
Continuous stable regularity

We prove an analytic version of the stable regularity lemma, which applies to functions $f : V \times W \to [0, 1]$ which are (k, δ)-stable, where (k, δ)-stability means that there do not exist $a_1, \ldots, a_k \in V$, $b_1, \ldots, b_k \in W$ with $|f(a_i, b_j) - f(a_j, b_i)| > \delta$ for all $1 \leq i < j \leq k$. This is joint work with Conant and Chavarria.

Sasha Razborov, University of Chicago [Friday 9.00 – 9.55]
Natural quasi-randomness

The theory of quasi-random (sometimes called pseudo-random) dense graphs is a well-developed theory with many important applications in and connections to quite different areas, both in combinatorics and beyond. There is no ambiguity in this definition: already the seminal papers by Chang, Graham, Wilson and Thomason addressed this notion from very different perspectives, and all attempts, both theirs and subsequent, have lead to equivalent results. The structural picture becomes much more interesting and elaborated already for hypergraphs and even more so for models of an arbitrary universal theory in a relational language. Indeed, there are many different notions of quasi-randomness in the literature, some of them forming hierarchies while others are incomparable. We contribute to this line of research by proposing quasi-randomness properties based upon several general principles that work uniformly for arbitrary universal first-order theories rather than tailored to a specific class of combinatorial objects (which is why we call these properties "natural"). Two other requirements are not to depend on ad hoc densities and to be closed under logical interpretations of an appropriate kind. A key concept in this theory is that of unique coupleability roughly meaning that any alignment of two objects on the same ground set should "look like" random. Joint work with Leonardo Coregliano.

Arturo Rodriguez Fanlo, University of Oxford [Wednesday 10.05 – 10.35]
Generics, closed approximate subgroups and a product theorem

I will present a relation between three different results: the first is a purely model-theoretic result on piecewise hyperdefinable groups; the second is Machado's Closed Approximate Subgroup Theorem; the last is Saxcé's Product Theorem for Simple Lie groups.

Jozsef Solymosi, University of British Columbia [Tuesday 11.45 – 12.25]
Geometric triples in the plane

The main question I'm going to talk about is: what can we say about the structure of planar pointsets containing many collinear triples? For a given set of points, P, a line is determined by P if it contains at least two points of P. If such a line contains exactly two points then it is called an *ordinary* line. Green and Tao proved a strong structure theorem which states that if P has at most Kn ordinary lines then all but $O(K)$ points of P lie on a cubic curve, if n is sufficiently large depending on K (in this result K is a constant or a very slowly growing function of n). Unfortunately almost nothing is known about the structure of P when there are cn^2 lines with at least three points of P, where $0 < c < 1/6$. Under some combinatorial conditions one can prove that if there are at least cn^2 collinear triples in P, then it has some structure. For example, if the number of distinct directions between many pairs of points of a pointset in convex position is small, then many points should lie on a conic. We use tools like regularity lemmas and Elekes-Szabo type results.

Terence Tao, University of California, Los Angeles [Monday 9.15 – 10.10]
Tutorial: *The circle method from the perspective of higher order Fourier analysis*
Higher order Fourier analysis is a collection of results and methods that can be used to control multilinear averages (such as counts for the number of four-term progressions in a set) that are out of reach of conventional linear Fourier analysis methods (i.e., out of reach of the circle method). One notable feature of this theory is that the role of linear phase functions is replaced by the notion of a nilsequence. On the other hand, key identities from linear Fourier analysis, such as the Plancherel identity or the Fourier inversion formula, are notably absent in the higher order theory. In this survey talk we give an introduction to the higher order Fourier theory by revisiting the linear circle method from a higher order perspective, in particular downplaying as much as possible the role of Fourier identities.

Caroline Terry, The Ohio State University [Monday 16.35 – 17.15]
Higher order stability and regularity

Matthew Tointon, University of Bristol [Wednesday 9.00 – 9.55]
Tutorial: *Approximate groups - history, basic properties, and some recent applications*
Approximate groups originated a number of years ago in additive combinatorics, and have since had a remarkably diverse range of applications. I will explain some of the history and motivation behind the notion; survey some basic properties of approximate groups; mention how model theory has contributed to their study; and finally describe some very recent progress and applications.

Henry Towsner, University of Pennsylvania [Tuesday 16.35 – 17.15]
Topology and hypergraph regularity
We pursue the link between hypergraph removal and Lebesgue points of density to give a general proof of induced hypergraph removal which immediately generalizes to ordered hypergraphs. We then use this result to show that the behavior of points of density can be used to characterize combinatorial properties like bounded VC_k-dimension.

Tingxiang Zou, Universität Münster [Tuesday 11.05 – 11.35]
The spatial orchard problem on cubic surfaces
One version of the famous orchard problem asks for arbitrarily large finite sets of size n in the plane with approximately n^2 collinear triples. One solution is the family of arithmetic progressions on a cubic curve. Furthermore, Elekes-Szabó proved any solution concentrating on an algebraic curve is of this form.

The spatial orchard problem asks for the same configuration but for sets of points in $\mathbb{R}^3$ or $\mathbb{C}^3$. It is natural to ask if there are solutions that are essentially non-planar. We consider whether finite subsets of projective smooth complex cubic surfaces would give answers to the spatial orchard problem. We showed that any such solution essentially comes from a planar cubic curve on the surface. The proof relies on the description of divisor classes on smooth cubic surfaces and various versions of Szemerédi-Trotter Theorem, in particular, we need the sharp bound of point-line incidences in $\mathbb{C}^2$ obtained by Tóth. This is a work-in-progress joint with Martin Bays and Jan Dobrowolski.