RAMSEY MULTIPLICITY OF LINEAR PATTERNS IN CERTAIN FINITE ABELIAN GROUPS

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ABSTRACT. It is well known (and a result of Goodman) that a random 2-colouring of the edges of a large complete graph K_n contains asymptotically (in n) the minimum number of monochromatic triangles (K_3s). Erdős conjectured that a random 2-colouring also minimises the number of monochromatic K_4s, but his conjecture was disproved by Thomason in 1989. The question of determining for which small graphs Goodman's result holds true remains wide open.

In this article we explore an arithmetic analogue: what can be said about the number of monochromatic additive configurations in 2-colourings of finite abelian groups? While we are able to answer several instances of this question using techniques from additive combinatorics and quadratic Fourier analysis, the main purpose of this paper is to advertise this sphere of problems and to put forward a number of concrete conjectures. We also note that, perhaps surprisingly, some of our results in the arithmetic setting have implications for the original graph-theoretic problem.

1. Graph-theoretic origins of the problem

Ramsey's theorem states that for any integer $s \geq 2$, there is a least integer n such that whenever we colour the edges of a complete graph K_n on n vertices red and blue, we are guaranteed to find a complete monochromatic subgraph of size s. Determining this least integer n, known as the *Ramsey number* and denoted $r(s)$, is a famously difficult open problem, with even the value of $r(5)$ currently unknown.

It stands to reason that when the number of vertices n in the complete graph K_n exceeds the required minimum $r(s)$, then we are not only guaranteed one monochromatic K_s, but many. Indeed, a simple double-counting argument shows that for fixed s and as $n \to \infty$, the number of monochromatic copies of K_s in any 2-colouring of the edges of K_n is $\Omega(n^s)$. Of course, in a random 2-colouring of K_n, where we assign the colour red to each edge independently at random with probability 1/2 and the colour blue otherwise, we would expect $2^{1-\binom{s}{2}}\binom{n}{s}$ such copies.

In order to be more precise we will need to introduce some notation. Let H and G be graphs, where we think of H as a small fixed graph and G a graph on a number of vertices tending to infinity. A homomorphic copy of H in G is a map $f : V(H) \to V(G)$ such that $f(u)f(v) \in E(G)$ whenever $uv \in E(H)$. We denote by $\hom(H, G)$ the number of homomorphic copies of H in G, and by $t_H(G) := \hom(H, G)/|V(G)|^{|V(H)|}$ its normalised

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companion, i.e. the probability that a randomly chosen map from $V(H)$ to $V(G)$ is a homomorphism. Let us also define the *Ramsey multiplicity* of H with respect to the colouring of the edges of $K_{|V(G)|}$ induced by G and its complement G^C as the quantity $m_H(G, G^C) := t_H(G) + t_H(G^C)$, where G^C is the graph defined on $V(G)$ such that uv is an edge in $E(G^C)$ if and only if $uv \notin E(G)$. Finally, we set $m_H(n) := \min_{G:|V(G)|=n} m_H(G, G^C)$ and $m_H := \lim_{n\to\infty} m_H(n)$.

It follows from the above discussion that for a graph G whose edges are chosen randomly with probability $1/2$ we have $m_{K_s}(G, G^C) = 2^{1-\binom{s}{2}} + o(1)$ with high probability, and thus $\Omega(1) \leq m_{K_s} \leq 2^{1-\binom{s}{2}}$.

Using an elementary argument (which can be simplified further), Goodman [12] proved in the 1950s that $m_{K_3} = 2^{1-\binom{3}{2}}$. Erdős [11] conjectured in 1962 that for all $s > 3$, it was indeed the random colouring that minimised the number of monochromatic copies of K_s amongst all 2-colourings of the edges of a large K_n. In other words, he conjectured that for every integer $s \geq 2$,

$$m_{K_s} = 2 \cdot \left(\frac{1}{2}\right)^{\binom{s}{2}}.$$

This conjecture was disproved by Thomason [35] in 1989, who exhibited a 2-colouring of an infinite family of K_ns which contained asymptotically strictly fewer than $2 \cdot \left(\frac{1}{2}\right)^6 \binom{n}{4}$ monochromatic K_4s. A few years prior Burr and Rosta [2] had in fact optimistically generalised Erdős's conjecture to every fixed graph H, positing that

$$m_H = 2 \cdot \left(\frac{1}{2}\right)^{|E(H)|}. \tag{1.1}$$

Clearly, since Erdős's conjecture is false, the Burr-Rosta conjecture does not hold for general graphs H. This was shown, independently of Thomason, by Sidorenko [29], who defined a sequence of edge-colourings of K_n that contained too few monochromatic copies of a graph H that consisted of a triangle with one additional edge attached to one of the vertices.

However, the Burr-Rosta conjecture has been verified for several classes of graphs. In keeping with the usual terminology, we shall call a graph H for which the Burr-Rosta conjecture (1.1) holds *common*. Graphs known to be common include trees, cycles, even-spoked wheels, triangular edge- and vertex-trees [29, 30, 22, 36], to name just a few. For a concise overview of known results and open problems in this area, including a weaker version of (1.1) that may hold for all graphs H, see Section 2.6 of the survey [6].

A particularly significant class of graphs that are known to be common stem from a closely related and arguably more acclaimed conjecture in graph theory, which has received a significant amount of attention lately (and also appeared in a slightly weaker form in [31]).

Conjecture 1.1 (Sidorenko's conjecture)**.** For every fixed bipartite graph H and every graph G, we have

$$t_H(G) \geq t_{K_2}(G)^{|E(H)|}.$$

Here of course $t_{K_2}(G)$ can be interpreted as (twice) the edge density $\delta(G)$ of G, where $\delta(G) := |E(G)|/\binom{|V(G)|}{2}$, so again the conjecture states that the minimum number of copies of H in G essentially occurs when G is a random graph.

We call a graph H satisfying Conjecture 1.1 *Sidorenko*. Clearly if a graph H is Sidorenko then it is common. It is easily seen that trees are Sidorenko, and as a result of recent work by Li and Szegedy [23], Szegedy [33] and Conlon et al. [5, 7], Sidorenko's conjecture is known for other large classes of bipartite graphs H. This includes bipartite graphs H which have at least one vertex connected to all vertices in the other part [5]. The most general known examples are not straightforward to describe, and we refer the reader to [7].

Needless to say, the problem of determining the minimum number of monochromatic copies of a small fixed subgraph has been generalised to more than two colours, but we shall concentrate on two colours only in what follows.

2. An arithmetic analogue

In this article we shall explore the following arithmetic analogue of the above sphere of problems. Throughout this article, we let Z be a finite abelian group. In some of our examples we shall take Z to be the cyclic group $\mathbb{Z}/N\mathbb{Z}$ for a sequence of primes $N \to \infty$, but even more frequently we shall consider the case where $Z = \mathbb{F}_p^n$ denotes a vector space of dimension n over a finite field of prime characteristic p.[1] In the 'finite-field model' as it is generally understood (see [17, 38]), it is important that p be thought of as small and fixed (we shall see $p = 3$ and 5 most frequently in the sequel), while the dimension n is to be thought of as tending to ∞. Asymptotic results for this group are thus asymptotic in n.

Let L be a system of m homogeneous linear equations in d variables with integer coefficients. Let A be a subset of Z, and let $\mathrm{sol}_{\mathrm{Z}}(L, A)$ denote the number of solutions to L in A. Define the *arithmetic multiplicity* of L in A by $t_{Z,L}(A) := \mathrm{sol}(L, A)/|Z|^{\deg(L)}$, where $\deg(L)$ is the number of degrees of freedom of the linear system L.[2] In other words, $t_{Z,L}(A)$ denotes the probability that a randomly chosen solution to L in Z forms a solution to L in A. Analogously to the graph-theoretic problem, we define the *arithmetic Ramsey multiplicity* of L with respect to the colouring induced by A and its complement A^C by $m_{Z,L}(A, A^C) := t_{Z,L}(A) + t_{Z,L}(A^C)$, where $A^C := Z \setminus A$.

For translation-invariant systems L, a generalised version of Szemerédi's theorem in finite abelian groups tells us that $m_{Z,L}(A, A^C) = \Omega(1)$ as $|Z| \to \infty$, so we shall almost exclusively focus our attention on translation-invariant systems.[3] Unlike the case of graphs, it seems to not be at all straightforward to show that the limit as $|Z| \to \infty$ of the minimum over

[1] Groups of non-prime order and fields of prime-power characteristic frequently introduce distracting divisibility restrictions, which we shall try to avoid here.

[2] The degree of L corresponds to the dimension of the solution space of L.

[3] The arithmetic analogue of Ramsey's theorem, namely Rado's theorem, was generalised to abelian groups by Deuber [10], and implies a positive proportion of monochromatic solutions to non-translation invariant systems such as Schur triples (see Example 3.2 below) in certain families of finite abelian groups.

$A \subseteq Z$ of $t_{Z,L}(A)$ (and hence $m_{Z,L}(A, A^C)$) exists, even for simple configurations L and specific families of finite abelian groups Z, see [8, 4]. We shall not pursue this matter here.

As in Section 1, we shall call a linear patterns defined by L *Sidorenko* in Z if it occurs with frequency at least α^d in every subset of Z of density α, i.e. if $t_{Z,L}(A) \geq \alpha^d$ for all $A \subseteq Z$ of density α. We shall call a configuration L *common* in Z if it occurs asymptotically with frequency at least $2(1/2)^d$ in any 2-colouring of the elements of the group Z, that is, if for any $A \subseteq Z$, $m_{Z,L}(A, A^C) \geq 2(1/2)^d + o(1)$ (as $|Z| \to \infty$). As before, it is easy to see that if a linear system L is Sidorenko, then it is common.

Let us discuss a first (and important) example to clarify these definitions.

Example 2.1 (Additive quadruples)**.** Let Z be any finite abelian group. Let L be an additive quadruple, that is, a solution to the single equation $x + y = z + w$ with 3 degrees of freedom, which we shall denote by AQ. We claim that AQ is Sidorenko in Z and hence common. To see this, observe that the number of solutions to $x + y = z + w$ in a subset $A \subseteq Z$ can be written as

$$\sum_{x,a,b \in Z} 1_A(x) 1_A(x + a) 1_A(x + b) 1_A(x + a + b),$$

where 1_A denotes the indicator function of A, which takes the value $1_A(x) = 1$ if $x \in A$, and 0 otherwise. We may rearrange this equation in order to apply the Cauchy-Schwarz inequality, which yields

$$\operatorname{sol}(AQ, A) = \sum_{a \in Z} \left(\sum_{x \in Z} 1_A(x) 1_A(x + a) \right)^2 \geq \left(\sum_{a \in Z} \sum_{x \in Z} 1_A(x) 1_A(x + a) \right)^2 \Big/ \left(\sum_{a \in Z} 1^2 \right).$$

But the sum in a and x can be computed as $|A|^2$, which implies that the number of solutions to L in A is at least $|A|^4/|Z|$. In other words, $t_{Z,AQ}(A) \geq \alpha^4$, and hence $m_{Z,AQ} \geq 2 \cdot (1/2)^4$, as required.

This example allows us to explain why we call the set-up above the arithmetic analogue of the graph-theoretic question formulated in Section 1. Additive quadruples in subsets of finite abelian groups are well known to be in direct correspondence with 4-cycles in graphs. Specifically, given a subset $A \subseteq Z$, which we shall assume to be symmetric in the sense that $A = -A := \{-a : a \in A\}$, we can construct the so-called *Cayley graph* $\Gamma = \Gamma(Z; A)$ as follows. Let $V(\Gamma) := Z$, and let uv be an edge in Γ if and only if $u - v \in A$. It can easily be seen that the number of 4-cycles in Γ corresponds precisely to the number of additive quadruples in A.

More generally, such a correspondence can be set up for other linear patterns but the limitations are twofold. First, not every graph is (isomorphic to) a Cayley graph generated by a symmetric subset of Z (in fact, almost none of them are). Secondly, not all linear configurations can be made to arise from graphs. For example, it is well known through attempts of proving Szemerédi's theorem for progressions of length 4 that for such a linear system a 3-uniform hypergraph is needed to represent its solutions.

We shall discuss the precise nature of the connection between the graph-theoretic problem and its arithmetic analogue further in Section 7. In Section 3 we ease into the problem with some relatively straightforward examples of additive configurations, including hypercubes and 3-term arithmetic progressions. In Section 4 we recall (and extend) the more complicated example of 4-term arithmetic progressions, studied by the second author in [37]. We examine to what extent adding free variables makes a given linear pattern uncommon (Section 5), and subsequently present some results concerning the (perhaps surprisingly) difficult case of linear patterns defined by one equation in an even number of variables (Section 6).

What we hope will spark the reader's interest when presented with this broad range of examples is the analysis of the underlying reason for which a linear pattern turns out to be common or uncommon. Unlike the case of graphs, where a multitude of cases has been studied but any attempt at classification appears extremely difficult, there is a strong structural theory for the arithmetic instance of the problem which allows us to be more systematic in our approach. However, we shall see that this theory turns up a range of reasons for the (un)commonality of a given linear pattern, which we believe shows that the problem is a difficult one even in the arithmetic setting. We present a summary of the emerging picture in Section 8.

3. Some straightforward patterns

We shall start out by discussing a simple generalisation of Example 2.1. For a definition of the Fourier transform, the Gowers uniformity norms and other standard notation, we refer the reader to [18].

Example 3.1 (Additive k-tuples)**.** Let Z be any finite abelian group. Let $k \geq 2$ be an integer and consider solutions to the equation $x_1+x_2+\cdots+x_k = x_{k+1}+x_{k+2}+\cdots+x_{2k}$. We shall denote the this pattern by AQ_k, and show that it is Sidorenko (and hence common). We give an alternative argument for the case $k=2$ using the Fourier transform. Note that

$$t_{Z,AQ_2}(A) = \mathbb{E}_{x,a,b} 1_A(x)1_A(x+a)1_A(x+b)1_A(x+a+b) = \mathbb{E}_x 1_A * 1_A(x)^2 = \sum_{\gamma} |\widehat{1_A}(\gamma)|^4,$$

which by positivity of the summand is bounded below by $|\widehat{1_A}(0)|^4 = \alpha^4$. More generally, it is easy to see that

$$t_{Z,AQ_k}(A) = \mathbb{E}_x 1_A * 1_A * \cdots * 1_A(x)^2 = \sum_{\gamma} |\widehat{1_A}(\gamma)|^{2k},$$

where the convolution is k-fold, so that $t_{Z,AQ_k}(A) \geq \alpha^{2k}$ as claimed.

What can we say about other linear patterns defined by one equation? We shall examine the first non-trivial case, namely that of an equation in three variables.

Example 3.2 (Schur triples)**.** Let Z be any finite abelian group. A *Schur triple*, denoted by ST, is a solution to the equation $x+y=z$. It is well known that there are dense sets containing no Schur triples at all, implying that this pattern is not Sidorenko. For example, in $\mathbb{F}_p^n$ we may take any non-trivial coset of a subspace of codimension 1, which has density $1/p$ in the group. In $\mathbb{Z}/N\mathbb{Z}$ it is easy to see that the "middle-thirds" set

$\{x \in \mathbb{Z}/N\mathbb{Z} : N/3 \leq x < 2N/3\}$ contains no solutions to $x + y = z$. A comprehensive analysis of the quantity $t_{Z,ST}(A)$ for various groups Z has recently been given by Samotij and Sudakov in [27].

On the other hand, it is elementary to see that Schur triples are common, see for example Theorem 1 in [9] and Corollary 3.1 in [3]. One way is to simply expand

$$m_{Z,ST}(A, A^C) = \mathbb{E}_{x,y} 1_A(x) 1_A(y) 1_A(x + y) + \mathbb{E}_{x,y} 1_{A^C}(x) 1_{A^C}(y) 1_{A^C}(x + y),$$

replacing 1_{A^C} by $1 - 1_A$, and evaluating the corresponding sums. Note that the terms containing a triple product of indicator functions disappear, due to a sign change. We obtain

$$m_{Z,ST}(A, A^C) = \alpha^3 + (1 - \alpha)^3 \geq \left(\frac{1}{2}\right)^2.$$

Alternatively, we may observe that the number of Schur triples in A may be written as

$$t_{Z,ST}(A) = \mathbb{E}_{x,y} 1_A(x) 1_A(y) 1_A(x + y) = \sum_\gamma \widehat{1_A}(\gamma)^2 \overline{\widehat{1_A}(\gamma)}.$$

Noting that $\widehat{1_A}(\gamma) = -\widehat{1_{A^C}}(\gamma)$ for $\gamma \neq 0$, we obtain that

$$m_{Z,ST}(A, A^C) = \widehat{1_A}(0)^3 + \widehat{1_{A^C}}(0)^3 = \alpha^3 + (1 - \alpha)^3 \geq \left(\frac{1}{2}\right)^2$$

as before.

It is remarkable that we not only obtain a lower bound on the number of monochromatic Schur triples, but in fact an exact formula for $m_{Z,ST}(A, A^C)$ whose value only depends on the density of the colour classes. An identical phenomenon occurs in the following example.

Example 3.3 (3-term arithmetic progressions)**.** Let $Z = \mathbb{Z}/N\mathbb{Z}$ or $Z = \mathbb{F}_p^n$ with $p > 2$. A *3-term arithmetic progression*, or $3 - AP$, is defined by the equation $x + y = 2z$. Giving a lower bound for $t_{3-AP}(A)$ in terms of the density of A corresponds to the infamously difficult problem of obtaining upper bounds in Roth's theorem (see for example Chapter 10 of [34]). In particular, it is known through recent work of Bloom [1] (building on prior work of Sanders [28]) that in $\mathbb{Z}/N\mathbb{Z}$,

$$t_{\mathbb{Z}/N\mathbb{Z},3-AP}(A) \geq \alpha^{O(\alpha^{-1}\log^3(\alpha^{-1}))},$$

a far cry from the expected α^3 in a random set. On the other hand, Green and Sisask [19] exhibited a subset A of $\mathbb{Z}/N\mathbb{Z}$ of density 1/2 for which $t_{\mathbb{Z}/N\mathbb{Z},3-AP}(A) = 5/48 < 1/8$, using ideas we shall return to in Section 5. It follows that 3-term arithmetic progressions are not Sidorenko in $\mathbb{Z}/N\mathbb{Z}$.

However, by the same reasoning as in Example 3.2, 3-APs are common. Both proofs adapt without difficulty to give

$$m_{Z,3-AP}(A, A^C) = \alpha^3 + (1 - \alpha)^3,$$

the Fourier approach relying on the identity $t_{Z,3-AP}(A) = \sum_\gamma \widehat{1_A}(\gamma)^2 \overline{\widehat{1_A}(\gamma^2)}$.

It was observed, by Cameron, Cilleruelo and Serra [3] for example, that the same argument can be extended to the analysis of any linear pattern defined by one equation in an odd number of variables. We leave the proof as an easy exercise to the reader.

Lemma 3.1 (One equation in an odd number of variables)**.** Let Z be any finite abelian group, and let OE be a linear pattern defined by one equation in an odd number of variables, i.e. $\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k = 0$ for some odd integer k and integers $\beta_1, \beta_2, \ldots, \beta_k$. Let A be a subset of Z of density α. Then

$$m_{Z,OE}(A, A^C) = \alpha^k + (1-\alpha)^k \geq \left(\frac{1}{2}\right)^{k-1},$$

from which it follows that OE is common.

Both proofs clearly fail, as a result of lack of cancellation, for even values of k. We shall return to the unexpectedly difficult case of one equation in an even number of variables in Section 6.

The next configuration, which combines two instances of Example 3.2 above, is inspired by a well-known example in graph theory, namely that of two triangles joined at a vertex (see Section 7).

Example 3.4 (Bow ties)**.** Let Z be any finite abelian group. We denote by BT the pattern defined by the simultaneous equations

$$\begin{array}{rcl} x+y &=& z \\ u+v &=& w \end{array}.$$

Note that since there is no overlap in the variables $t_{Z,BT}(A) = t_{Z,ST}(A)^2$, and hence

$$m_{Z,BT}(A, A^C) = t_{Z,ST}(A)^2 + t_{Z,ST}(A^C)^2 \geq \frac{1}{2}(t_{Z,ST}(A) + t_{Z,ST}(A^C))^2,$$

which equals

$$\frac{1}{2} m_{Z,ST}(A, A^C)^2 \geq 2 \cdot \left(\frac{1}{2}\right)^6$$

as desired.

More generally, combinations of (the same) common pattern using mutually disjoint sets of variables are easily seen to be common as well.

We conclude this section by generalising Example 2.1 in another direction, which belongs to a different "complexity class" from the above, and will naturally lead us on to Section 4.

Example 3.5 (Hypercubes)**.** Let Z be any finite abelian group, and let $d \geq 2$ be an integer. We call the linear configuration C_d given by the 2^d linear forms $(x_0 + \sum_{i=1}^d \epsilon_i x_i)_{\epsilon \in \{0,1\}^d}$ a *hypercube of dimension* d. It can easily be verified that when $d = 2$ this definition again reduces to that of an additive quadruple.

Just like an additive quadruple, the hypercube of dimension d is Sidorenko (and hence common) for $d > 2$. The proof in Example 2.1 using the Cauchy-Schwarz inequality generalises easily, and allows us to prove by induction that $t_{Z,C_d}(A) \geq \alpha^{2^d}$ and hence $m_{Z,C_d}(A, A^C) \geq (1/2)^{2^d-1}$.

4. Arithmetic progressions of length 4

The reader may care to verify that the preceding example (Example 3.5) does not have a Fourier-based proof of commonality as for $d > 2$ there is no useful expression for the number of hypercubes in terms of the Fourier transform. The latter fact is also true of 4-term arithmetic progressions, and it is for this reason that Gowers introduced the uniformity norms (see, for example, Definition 2.12 in [38]).

Arithmetic progressions of length 4, denoted by $4-AP$ and defined by the equations $x + y = 2z$ and $y + z = 2w$, are clearly not Sidorenko, and finding lower bounds on $t_{Z,4-AP}(A)$ in terms of the density of A corresponds to finding good bounds in Szemerédi's theorem in the group Z.

Motivated by Thomason's proof [35] that the graph K_4 is uncommon, the second author showed in [37] that 4-term arithmetic progressions are uncommon in $\mathbb{Z}/N\mathbb{Z}$. Specifically, it was shown that there exists a set $A \subseteq \mathbb{Z}/N\mathbb{Z}$ for which

$$m_{\mathbb{Z}/N\mathbb{Z},4-AP}(A, A^C) < \left(1 - \frac{1}{259200}\right) \times \left(\frac{1}{2}\right)^3 \approx 0.12499952,$$

where $(1/2)^3$ is of course the proportion of 4-APs expected in a random set. In subsequent work Lu and Peng [24] improved the right-hand side to the much more reasonable $68/75 \times (1/2)^3 \approx 0.113333$.[4]

Given that 4-term arithmetic progressions are uncommon in $\mathbb{Z}/N\mathbb{Z}$, it is natural to ask the following question.

Question 4.1. For a given finite abelian group Z, what is $\min_{A\subseteq Z} m_{Z,4-AP}(A, A^C)$?

In [37] it was shown that

$$\min_{A\subseteq\mathbb{Z}/N\mathbb{Z}} m_{\mathbb{Z}/N\mathbb{Z},4-AP}(A, A^C) \geq \left(\frac{1}{2}\right)^4,$$

and the right-hand side was improved by Lu and Peng [24] to 7/96. In graph theory the corresponding minimisation problem has been studied extensively. For the best known upper and lower bounds see [36] and [32], respectively.

We use this section to add three further observations to the existing body of work on 4-term arithmetic progressions. First, we shall give a finite-field version of the construction in [37], which has the benefit of being significantly easier to understand, and requiring no strenuous computation whatsoever. Secondly, we examine what can be said about the structure of those colourings that show 4-APs to be uncommon in this setting. Thirdly, we analyse to what extent these methods can be used to obtain results about configurations containing 4-term progressions.

[4] Here and elsewhere in the paper, numerical results are given to eight significant figures.

The first colouring containing fewer than the expected number of monochromatic 4-term progressions in [37] was based on an unpublished construction from quadratic Fourier analysis due to Gowers [14], who had constructed a subset of $\mathbb{Z}/N\mathbb{Z}$ of density $1/2$ which was uniform in the sense that the non-trivial Fourier coefficients of its indicator function were small, but which contained significantly fewer than the expected number of 4-APs.[5] In $\mathbb{F}_5^n$, many of the technical details simplify and we give them in full here.

Example 4.1 (4-term arithmetic progressions)**.** There exists a set $A \subseteq \mathbb{F}_5^n$ such that

$$m_{\mathbb{F}_5^n,4-AP}(A,A^C) \leq \frac{1}{8} - \frac{7}{2^{10}5^2} \approx 0.12472656.$$

We shall break up the construction of A into a number of claims.

Claim 4.2. Let $f : \mathbb{F}_5 \to \{-1,1\}$ be defined by $f(x) = -1$ if $x = 0$ and $f(x) = 1$ otherwise. Then

$$\mathbb{E}_{x,d\in\mathbb{F}_5} f(x)f(x+d)f(x+2d)f(x+3d) = -\frac{7}{25}.$$

Moreover, if we let $V := \mathbb{F}_5^{n-1}$ so that $\mathbb{F}_5^n = V \oplus V^\perp$, and define $F : \mathbb{F}_5^n \to \{-1,1\}$ by setting $F(x) := f(y)$ when $x \in V + y$ for $y \in V^\perp$, then

$$\mathbb{E}_{x,d\in\mathbb{F}_5^n} F(x)F(x+d)F(x+2d)F(x+3d) = -\frac{7}{25}.$$

Proof. There are 25 4-term progressions in $\mathbb{F}_5$, including 5 trivial ones which contribute $5/25$ to the expectation in f. Each non-trivial progression is counted 4 times, and all besides $1,2,3,4$ contribute $-4/25$ to the expectation, amounting to a total contribution of $-16/25$. The progression $1,2,3,4$ contributes $4/25$, for a total of $(5-16+4)/25 = -7/25$. The second part of the claim is immediate by splitting $\mathbb{E}_{x\in\mathbb{F}_5^n} = \mathbb{E}_{y\in V^\perp}\mathbb{E}_{v\in V}$, and similarly for d. □

Claim 4.3. Let $G : \mathbb{F}_5^n \to [-4,4]$ be defined by $G(x) := F(x)(\omega^{x\cdot x} + \omega^{-3x\cdot x} + \omega^{3x\cdot x} + \omega^{-x\cdot x})$. Then $|\widehat{G}(t)| = o(1)$ for all $t \in \mathbb{F}_5^n$. Moreover,

$$|\mathbb{E}_{x,d\in\mathbb{F}_5^n} G(x)G(x+d)G(x+2d)G(x+3d) - 4\mathbb{E}_{x,d\in\mathbb{F}_5^n} F(x)F(x+d)F(x+2d)F(x+3d)| = o(1).$$

Proof. It is easily computed that for any affine subspace $W = w + W_0$, where $W_0 \leqslant \mathbb{F}_5^n$, $|\widehat{1_W}(t)| = 1/|W_0^\perp|$ if $t \in W_0^\perp$, and 0 otherwise. Recall also that from standard Gauss sum estimates, $|\widehat{\omega^q}(t)| = o(1)$ for any $t \in \mathbb{F}_5^n$ and any quadratic form q of rank tending to infinity with n. Thus for any affine subspace W, the function $G'(x) = 1_W(x)\omega^{q(x)}$ has Fourier transform of size

$$|\widehat{G'}(t)| = |\widehat{1_W} * \widehat{\omega^q}(t)| = |\sum_s \widehat{1_W}(t-s)\widehat{\omega^q}(s)| \leq \sup_s |\widehat{\omega^q}(s)| \sum_s |\widehat{1_W}(s)| = \sup_s |\widehat{\omega^q}(s)| = o(1).$$

In order to obtain the result for G, it remains to observe that F is a plus/minus one combination of indicator functions of 5 affine subspaces.

[5] It is significantly easier to obtain an example of a uniform set containing significantly *more* than the expected number of 4-APs, see for example Section 2.3 of [38].

To see why the second part of the claim is true we have to work a tiny bit harder. We start by expanding the product

$$\mathbb{E}_{x,d\in\mathbb{F}_5^n} G(x)G(x+d)G(x+2d)G(x+3d)$$

into $4^4 = 256$ terms, each of which is of the form

$$\mathbb{E}_{x,d\in\mathbb{F}_5^n} F(x)F(x+d)F(x+2d)F(x+3d)\omega^{ax\cdot x+b(x+d)\cdot(x+d)+j(x+2d)\cdot(x+2d)+k(x+3d)\cdot(x+3d)},$$

where each of a, b, j, k takes one of the values $+1$,-3,3 or -1. It can easily be checked that the only assignments (a, b, j, k) that leave the exponent equal to zero are $(1, -3, 3, 1)$, $(-1, 3, -2, 1)$, $(3, 1, -1, -3)$ and $(-3, -1, 1, 3)$. Together these four contributions give rise to a term of the form

$$4\mathbb{E}_{x,d\in\mathbb{F}_5^n} F(x)F(x+d)F(x+2d)F(x+3d).$$

All remaining terms, which involve a product of copies of F with a quadratic exponential or a non-trivial bilinear phase in x and d, are negligible by a variant of the argument made for G' at the start of the proof. □

Claim 4.4. Let $h : \mathbb{F}_5^n \to [0, 1]$ be defined by $h(x) = \frac{1}{8}(G(x)+4)$. Then $\mathbb{E}_x h(x) = \frac{1}{2}+o(1)$ and $|\widehat{h}(t)| = o(1)$ for all $t \neq 0$. Moreover,

$$\mathbb{E}_{x,d\in\mathbb{F}_5^n} h(x)h(x+d)h(x+2d)h(x+3d) \leq \frac{1}{16} - \frac{7}{2^{10}5^2} + o(1).$$

Proof. The claims concerning the average and the Fourier coefficients of h are easy to verify. To see the final inequality, expand the product

$$\mathbb{E}_{x,d\in\mathbb{F}_5^n} h(x)h(x+d)h(x+2d)h(x+3d)$$

into 16 terms. The term arising from having chosen 4 from each bracket gives the main contribution of $(1/2)^4 = 1/16$. Combining Claims 4.2 and 4.3, we see that the term

$$\frac{1}{2^{12}}\mathbb{E}_{x,d\in\mathbb{F}_5^n} G(x)G(x+d)G(x+2d)G(x+3d)$$

arising from having chosen G from each bracket contributes

$$\leq -\frac{4\times 7}{25}\frac{1}{2^{12}} + o(1).$$

All other terms are negligible as they define configurations in G consisting of at most 3-terms, all of which are controlled by the Fourier coefficients of G. □

The function h can now be converted into a subset of $\mathbb{F}_5^n$ by a standard probabilistic argument, namely by letting $x \in \mathbb{F}_5^n$ lie in the desired set with probability $h(x)$. We leave the details to the reader.

This concludes the proof of the example. In contrast to this analytic way of proceeding, Lu and Peng [24] used a brute-force computational approach in $\mathbb{Z}/N\mathbb{Z}$ which turned out to be quantitatively superior.[6] However, the following simple lemma shows that, at least in

[6]Such an approach can also be implemented in $\mathbb{F}_5^n$, but we are not interested in obtaining the best constants here.

a weak sense, any colouring which contains fewer than the expected number of monochromatic 4-term progressions (which we shall refer to as *bad* in the sequel) must arise from a quadratically structured example as in [37].

Proposition 4.5 (Structure of bad colourings for 4-APs)**.** Let $0 < \delta < 1/8$ and suppose that $A \subseteq \mathbb{F}_5^n$ is such that

$$m_{\mathbb{F}_5^n,4-AP}(A, A^C) < \left(\frac{1}{2}\right)^3 - \delta.$$

Then there exists a quadratic form q on $\mathbb{F}_5^n$ such that $|\mathbb{E}_x 1_A(x)\omega^{q(x)}| \geq c'(\delta)$, where c' is a function that tends to zero as δ tends to zero.

Proof. Note that by definition, assumption and convexity,

$$m_{\mathbb{F}_5^n,4-AP}(A, A^C) = t_{\mathbb{F}_5^n,4-AP}(A) + t_{\mathbb{F}_5^n,4-AP}(A^C) < \left(\frac{1}{2}\right)^3 - \delta \leq \alpha^4 + (1-\alpha)^4 - \delta,$$

where α denotes the density of A in $\mathbb{F}_5^n$ as usual. In particular, at least one of

$$|t_{\mathbb{F}_5^n,4-AP}(A) - \alpha^4| \geq \delta/2 \quad \text{and} \quad |t_{\mathbb{F}_5^n,4-AP}(A^C) - (1-\alpha)^4| \geq \delta/2$$

must hold. Now let $f_A := 1_A - \alpha$ be the so-called *balanced function* of A. In his work on Szemerédi's theorem for progressions of length 4, Gowers [13] showed[7] that

$$|t_{\mathbb{F}_5^n,4-AP}(A) - \alpha^4| \leq 5\|f_A\|_{U^3}.$$

An analogous statement holds for A^C, namely

$$|t_{\mathbb{F}_5^n,4-AP}(A^C) - (1-\alpha)^4| \leq 5\|f_A\|_{U^3},$$

where we have used the fact that $f_{A^C} = -f_A$. It follows that if the count of monochromatic 4-APs is less than expected by an amount δ, then we must have

$$\|f_A\|_{U^3} \geq 5\delta/2.$$

By the inverse theorem for the U^3 norm (see Theorem 2.3 in [20], which is based on prior work of Gowers [13]), we conclude that there exists a quadratic form q on $\mathbb{F}_5^n$ such that

$$|\mathbb{E}_x 1_A(x)\omega^{q(x)}| \geq c(5\delta/2),$$

where $c(\delta)$ is a constant that tends to 0 as δ tends to 0. □

At the time of writing we only know that $c(\delta)$ can be taken to be $\delta^{O(\log^C \delta^{-1})}$ for some constant C, although it is conjectured to be polynomial in δ (see, for example, Section 4.4 of [38]). Hence Proposition 4.5 is primarily of qualitative interest.[8] Having said this, we believe it provides the first insight into the structure of bad colourings. In particular, to our knowledge no such result is known for the original graph-theoretic problem (although interestingly, Thomason's first construction in [35] was also based on a quadratic form).

[7]Rather, he showed that the inequality holds in $\mathbb{Z}/N\mathbb{Z}$, but the argument is identical.

[8]A similar argument can be made in the case of $\mathbb{Z}/N\mathbb{Z}$, but since the statement of the U^3 inverse theorem is less clean in that setting we omit the details.

Finally, we note that any example of the kind constructed above also gives rise to a bad colouring for 5-APs. Similarly to [37], we can write

$$m_{\mathbb{F}_5^n,5-AP}(A,A^C)=\mathbb{E}_{x,d}\prod_{j=0}^{4}1_A(x+jd)+\mathbb{E}_{x,d}\prod_{j=0}^{4}1_{A^C}(x+jd)$$

as

$$-\sum_{i=0}^{4}t_{\mathbb{F}_5^n,5-AP(i)}(A)+\sum_{\{i,k\}\in\{0,1,2,3,4\}^{(2)}}t_{\mathbb{F}_5^n,5-AP(i,k)}(A)-\binom{5}{2}\alpha^2+5\alpha-1,$$

where we have written $t_{\mathbb{F}_5^n,5-AP(i)}(A)$ for the expression $\mathbb{E}_{x,d}\prod_{j=0}^{4}f_j(x+jd)$ where $f_j=1$ when $j=i$, and $f_j=1_A$ otherwise, and $t_{\mathbb{F}_5^n,5-AP(i,k)}(A)$ for the expression $\mathbb{E}_{x,d}\prod_{j=0}^{4}f_j(x+jd)$ where $f_j=1$ when $j=i$ or $j=k$, and $f_j=1_A$ otherwise. Notice that since we are working over $\mathbb{F}_5$, the configurations defined by $5-AP(i)$ are still 4-term progressions, while the configurations defined by $5-AP(i,k)$ are all 3-term progressions.[9]

Writing further $d_4=t_{\mathbb{F}_5^n,4-AP}(A)-\alpha^4$ and $d_3=t_{\mathbb{F}_5^n,3-AP}(A)-\alpha^3$ for the deviation from the expected number, we find after some rearranging that

$$m_{\mathbb{F}_5^n,5-AP}(A,A^C)=\alpha^5+(1-\alpha)^5-5d_4+10d_3.$$

It follows as in [37] that a set A which is uniform (implying that $d_3=o(1)$) and which contains fewer than the expected number of 4-APs (meaning $d_4\leq -c$ for some positive constant c) gives rise to a colouring that contains fewer than the expected number of 5-term arithmetic progressions. Again, cancellation has come to the rescue.

While perhaps not entirely unexpected, the fact that the bad colouring for 4-APs is also bad for 5-APs bears emphasising as we had no a priori information about the number of 5-APs in A. It also illustrates the power of the quadratic Fourier analysis approach, as the purely computational one would have required us to start our calculations again from zero. The following question naturally arises.

Question 4.6. Is it true that every linear configuration containing a 4-AP is uncommon?

While we expect the answer to be positive, it does not seem to follow immediately from the above observations. In graph theory the analogous result is known [22]: any graph containing a K_4 is uncommon.

5. Free variables skew densities

We continue our exploratory journey through the arithmetic forest, returning to a much simpler configuration. Again, the authors were inspired by an example in graph theory (see Section 7) when considering the slightly odd-looking set-up below, in which we have an unconstrained variable.

[9]This is not true in $\mathbb{Z}/N\mathbb{Z}$, but the argument that follows can be adapted. We leave the details to the energetic reader.

Theorem 5.1. *Let $Z = \mathbb{F}_3^n$, and let TP be the set of solutions (x, y, z, w) to the equation $x + y = 2z$. Then there exists $A \subseteq \mathbb{F}_3^n$ such that*

$$m_{\mathbb{F}_3^n,TP}(A, A^C) \approx 0.12463884 < 2\left(\frac{1}{2}\right)^4.$$

In other words, TP is uncommon in $\mathbb{F}_3^n$.

Proof. We take eight linearly independent vectors u, v_1, v_2, w_1, w_2, y_1, y_2, y_3, and let U, V, W and Y be subspaces of codimension 1, 2, 2 and 3, respectively, whose orthogonal complements are spanned by the correspondingly labelled vectors. Let A be the union of U, V, W and Y. It is not too difficult to compute, using inclusion-exclusion, that

$$|A| \approx 0.49276024|\mathbb{F}_3^n|$$

and with considerably more effort that

$$m_{\mathbb{F}_3^n,TP}(A, A^C) \approx 0.12463884,$$

as desired. The latter calculation can be carried out in any number of ways: by brute-force computation; using the Fourier-coefficients of the function 1_A, which are reasonably straightforward to write down; or carefully counting the number of 3-APs by hand, using inclusion-exclusion. We leave the details to the interested reader. □

Much more interesting than the construction itself is the sequence of observations that led to it, which we shall briefly record here. It follows from the definitions that for any set $A \subseteq \mathbb{F}_3^n$

$$m_{\mathbb{F}_3^n,TP}(A, A^C) = \alpha t_{\mathbb{F}_3^n,3-AP}(A) + (1-\alpha)t_{\mathbb{F}_3^n,3-AP}(A^C),$$

which, since $t_{\mathbb{F}_3^n,3-AP}(A) + t_{\mathbb{F}_3^n,3-AP}(A^C) = \alpha^3 + (1-\alpha)^3$, means that

$$m_{\mathbb{F}_3^n,TP}(A, A^C) = \alpha^4 + (1-\alpha)^4 + (2\alpha - 1)(t_{\mathbb{F}_3^n,3-AP}(A) - \alpha^3). \tag{5.1}$$

This immediately tells us that we are guaranteed to get the count of configurations expected in the random case whenever the set A is of density $1/2$. Therefore, if we wish to show that the configuration TP is uncommon we need to look for sets whose density differs from $1/2$ very slightly, as any large deviation will ensure that the term $\alpha^4 + (1-\alpha)^4$ takes control, undermining any hope of obtaining a lower than expected count. In addition, note that $m_{\mathbb{F}_3^n,TP}(A, A^C)$ depends on the deviation of the 3-AP count in A from the expected value. We are therefore led to searching for a set whose density is slightly below $1/2$ but which contains many more than the expected number of 3-APs. An obvious candidate for the latter is a subspace, and in Theorem 5.1 the codimensions were simply chosen so as to bring the density of the union as close to $1/2$ as possible.

These remarks also immediately lead us to an analogue of Theorem 5.1 in $\mathbb{Z}/N\mathbb{Z}$. Indeed, Green and Sisask [19] constructed for any $1/3 < \alpha < 2/3$ a set $A \subseteq \mathbb{Z}/N\mathbb{Z}$ (consisting of a union of arithmetic progressions, the $\mathbb{Z}/N\mathbb{Z}$-analogue of subspaces) of density α with the property that

$$t_{\mathbb{Z}/N\mathbb{Z},3-AP}(A) \leq \frac{2 - 12\alpha + 21\alpha^2}{12}.$$

An optimisation leads to a set $A \subseteq \mathbb{Z}/N\mathbb{Z}$ of density $\alpha \approx 0.50693243$ such that

$$m_{\mathbb{Z}/N\mathbb{Z},TP}(A, A^C) \approx 0.12485549 < 2\left(\frac{1}{2}\right)^4.$$

More can be said on the basis of Equation (5.1). Notice that if $m_{\mathbb{F}_3^n,TP}(A, A^C) < 2(1/2)^4 - \delta$ for some constant $0 < \delta < 1/8$, then the 3-AP count of A must deviate significantly from its expectation. Using standard Fourier-analytic arguments from the proof of Meshulam's theorem [25], for example, it can be shown that in this case there exists an element $t \in \mathbb{F}_3^n$, $t \neq 0$, such that $|\mathbb{E}_x 1_A(x)\omega^{x \cdot t}| \geq \delta/\alpha(1-2\alpha)$. From this it follows, again via a routine argument, that the set A must have linear structure in the sense that it is strongly biased towards a very large (potentially affine) subspace. Indeed, it is precisely a set with this property which gave us the construction in Theorem 5.1 in the first place. Our conclusions here are again merely of a qualitative nature, but should be compared with the structural information in Proposition 4.1, which stated that a bad colouring for 4-APs must be quadratically (and not just linearly) structured. This means that configurations can be uncommon for at least two genuinely distinct reasons, a point which we shall return to in Section 8.

It is of course possible to add unconstrained variables to other configurations, and we shall see another example in the final section of this article. To conclude this section, we record a natural question posed to us by Noga Alon.

Question 5.2. Is it true that adding sufficiently many free variables makes any configuration uncommon?

As before, a suitable version of such a statement is true for graphs (see Theorem 4 in [22]).

6. One equation in an even number of variables

We return to the case of configurations defined by a single equation. In Section 3 we established that for an odd number of variables, the configuration is not only always common, but that in fact an exact formula for the number of monochromatic solutions can be given in terms of the density of the colour classes. An identical argument exploiting cancellation, written down by Cameron, Cilleruelo and Serra in [3], shows that in the case of an even number of variables, the *difference* between the number of red solutions and the number of blue solutions is given by an exact formula in terms of the respective colour densities. When the colours are exactly balanced, one concludes that the number of red and the number of blue configurations is in fact the same.

However, the formula for the difference rather than the sum of monochromatic solutions does not address the question of whether such a configuration is common, and indeed the case of one equation in an even number of variables remains one of the most mysterious.

For simplicity we shall initially restrict our attention to translation-invariant[10] equations over $\mathbb{F}_5$ in 4 variables. It does not take long to check that there are only four genuinely distinct configurations.

(1) The *additive quadruple*, which we already encountered in Section 2. It is denoted by AQ and given by the equation $x + y = z + w$.
(2) The *heavy quadruple*, denoted by HQ. It is given by the equation $x + 2y = z + 2w$.
(3) The *heavy cycle*, denoted by HC. It is given by the equation $x + y + z = 3w$.
(4) The *skew quadruple*, denoted by SQ. It is given by the equation $2x + 2y = 3w + z$.

We have already seen in Section 2 that AQ is common. By an identical argument, so is the heavy quadruple HQ. What about the remaining two configurations?

Theorem 6.1. *Both HC and SQ are uncommon in $\mathbb{F}_5^n$.*

Proof. It turns out that the same construction works for both configurations. Let $V \leqslant \mathbb{F}_5^n$ be a subspace of codimension 1. Let A be the union of the $+1$ and -1 cosets of V in $\mathbb{F}_5^n$, together with half the elements of V chosen at random. Then with high probability the density of A is 1/2, and it is verified (in a rather tedious manner) that

$$m_{HC}(A, A^C) = 0.105 < 2 \cdot \left(\frac{1}{2}\right)^4 .$$

Indeed, the calculation reduces to showing that there are comparatively few solutions in $\{+1, -1\}$ to $x + y + z = 3w$ and $2x + 2y = 3w + z$, respectively. Since it is imperative that the density of A be (at least close to) 1/2, we add half of the trivial coset at random, which keeps the total solution count below the expected value. Moreover, the same holds for A^C by the remarks in the introductory paragraph of this section. □

By an argument similar to that outlined at the end of Section 5, it can be shown that a bad colouring for either of the above two configurations must have a large Fourier coefficient. This means that the colour distribution must exhibit a strong linear bias, i.e. be concentrated on a coset (or several) of a low-codimensional subspace. So again, in a qualitative sense the construction in Theorem 6.1 incorporates the true reason for HC and SQ being uncommon.

It is crucial at this point to try and understand what distinguishes AQ and HQ from HC and SQ. Clearly the former two equations exhibit some symmetry that is absent in the latter two. To make this precise we turn to a definition due to Ruzsa [26].[11]

Definition 6.2 (Genus). Let $Z = \mathbb{F}_p^n$, and let $m \geq 2$ and $g \geq 1$ be integers. A translation-invariant equation of the form

$$b_1x_1 + b_2x_2 + \cdots + b_mx_m = 0$$

with variables $x_i \in \mathbb{F}_p^n$ and integer coefficients b_i is said to have genus g *over* $\mathbb{F}_p$ if g is the largest integer such that there is a partition of $\{1, 2, \ldots, m\}$ into g disjoint non-empty

[10]Note that the notion of translation invariance depends on the characteristic of the underlying field.
[11]Ruzsa defined it over the integers, but we use it over a field $\mathbb{F}_p$ here.

subsets $I_1, I_2, \ldots, I_g$ with the property that

$$\sum_{i \in I_j} b_i = 0$$

for every $j = 1, 2, \ldots, g$.

We immediately point out that of course this definition of genus is, just like the notion of translation invariance itself, dependent on the characteristic. It is easy to see that AQ and HQ have genus 2 over $\mathbb{F}_5$ (and any field of larger characteristic), while HC and SQ have genus 1. After a little thought this observation leads to the following conjecture.

Conjecture 6.3. Let $k \geq 2$ be an integer. A linear configuration given by a single equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_kx_k = a_{k+1}x_{k+1} + a_{k+2}x_{k+2} + \cdots + a_{2k}x_{2k} \tag{6.1}$$

is common in $\mathbb{F}_p^n$ if and only if it has genus k over $\mathbb{F}_p$.

In one direction Conjecture 6.3 is easily seen to be true. Indeed, both the Fourier-argument in Example 3.1 and the Cauchy-Schwarz argument in Example 3.5 generalise to yield the result that any equation of genus k of the form (6.1) defines a common configuration. To test the reverse direction, we numerically investigated the case of translation-invariant equations in six variables over $\mathbb{F}_5$, of which only five are genuinely distinct.

$$x + y + z = u + v + w \tag{6.2}$$
$$x + y + 2z = u + v + 2w \tag{6.3}$$
$$x + y + z + w = 2u + 2v \tag{6.4}$$
$$x + y + z + w + 2u = v \tag{6.5}$$
$$x + y + z + 2w + 2u = 2v \tag{6.6}$$

The first two equations, (6.2) and (6.3), have genus 3 over $\mathbb{F}_5$, the remaining three equations have genus 2.[12] By the discussion above, configurations defined by either of the first two equations are therefore clearly common.

Example 6.1 (Genus < 3 in six variables over $\mathbb{F}_5$)**.** The configurations defined by any of the equations (6.4), (6.5) and (6.6) above are uncommon over $\mathbb{F}_5$. To see this for (6.5) and (6.6), use the cosets $1 + V$ and $-1 + V$ of a subspace $V \leqslant \mathbb{F}_5^n$ of codimension 1 as before, and add half the elements of V independently at random. For (6.4) we use $1 + V$ and $-2 + V$ instead.

Since there are no equations of genus 1 in six variables over $\mathbb{F}_5$, we also tested all fourteen translation-invariant equations in six variables over $\mathbb{F}_7$, of which five are of genus 3, eight are of genus 2 and one is of genus 1. All our results are consistent with Conjecture 6.3 above.

[12] Note that only equation (6.4) has genus 2 over the integers.

7. Relationship between sets and graphs

While we have emphasised the analogy between graph and arithmetic setting throughout, and briefly discussed the use of the Cayley graph in Section 2, it will be necessary for our final section to examine the extent to which results translate rigorously from one context to another.

The Cayley graph construction can be used to transfer a bad colouring from the arithmetic to the graph setting. This only works in a straightforward way, however, if

- the bad colouring found in the arithmetic case is symmetric, that is, if x is coloured red then so is $-x$; and
- the coefficients in the linear system are all equal to plus or minus 1.

In other words, if we can show that a certain linear configuration is uncommon under the above conditions, then we can also show that the corresponding graph is uncommon. We illustrate this idea with an example.

Example 7.1 (Triangle with a pendant edge)**.** The construction in Theorem 5.1 gives a new proof that the graph T' consisting of a triangle with a pendant edge is uncommon, a fact originally proved by Sidorenko [29].

Note that since the set $A \subseteq \mathbb{F}_3^n$ in Theorem 5.1 is defined as a union of subspaces, it is symmetric. It follows that we can define a Cayley graph $\Gamma = \Gamma(Z; A)$ on vertex set $V(\Gamma) = \mathbb{F}_3^n$ with uv being an edge if and only if $u - v \in A$. Any quadruple $(x, y, z, w) \in A^4$ satisfying $x + y + z = 0$ thus corresponds to a triangle uv, vs, us with a pendant edge ut in Γ, which can be seen by setting $x = u - v$, $y = v - s$, $z = s - u$ and $w = u - t$.

In fact, it is interesting to compare the structure of our colouring with that of Sidorenko's (who obtained a better constant).

In the other direction, it is not difficult to convince oneself that any proof showing that a given graph H is common can be adapted to show that an associated linear configuration is common. The reason is that essentially all known such proofs are based on the Cauchy-Schwarz inequality. We invite the reader to consider the following example.

Example 7.2 (Diamond)**.** A *diamond* is a graph consisting of two triangles identified at an edge. One of a number of possible associated linear configurations is the pattern D, defined by the simultaneous equations

$$\begin{aligned} x + y &= 2z \\ u + v &= 2z, \end{aligned}$$

consisting of two 3-term arithmetic progressions sharing a midpoint. The same calculation that shows that the diamond graph is common also shows that the linear configuration D is common in any finite abelian group Z. We leave the details as an exercise. As a consequence of Lemma 3.1, a similar argument works for any system of two identical equations in an odd number of variables, having one variable (with the same coefficient) in common.

Letting known results from graph theory guide us in this way, a relatively large number of linear systems can be shown to be common. It is impossible to give an exhaustive list, but systems of equations associated with triangular edge- or vertex-trees, or those associated with square wheels and other regular grid structures (see Chapter II of [21]), fall in this category. All of them have the property that they look very "regular", which makes it possible to apply an (often rather intricate) Cauchy-Schwarz argument.

8. Towards an arithmetic Burr-Rosta conjecture

Having examined a reasonable number of examples in the preceding sections, we have reached a point at which we might like to formulate a more general conjecture regarding which linear configurations are common and uncommon, respectively. Let us first summarise the different behaviours we have seen so far. We have witnessed that a linear configuration can be

(1) common because of cancellation (3-APs);
(2) common because of symmetry (AQ, HQ);
(3) uncommon because of skewed density (TP);
(4) uncommon because of pure linear bias (HC, SQ);
(5) uncommon for quadratic reasons (4-APs).

Any reasonable conjecture concerning the classification of linear patterns as common or uncommon must take into account all of these possibilities. Even though the analogy between the arithmetic and the graph setting is not perfect (see Section 7), this diverse array of underlying reasons may go some way towards explaining why the analogous graph-theoretic problem remains wide open despite having been studied for many years.

We may dig deeper yet into the analysis of the cut-off between behaviours (4) and (5) in the case where the linear pattern is defined by more than one equation. In order to do so, we recall a definition from [15].

Definition 8.1 (Square independence)**.** A linear configuration defined by linear forms $L_1, L_2 \ldots, L_m$ in d variables with integer coefficients is said to be *square independent over* $\mathbb{F}_p$ if the quadratic forms $L_i^T L_i$, $i = 1, 2, \ldots, m$, are linearly independent over $\mathbb{F}_p$.

In [15, 16] Gowers and the second author proved that a linear configuration defined by linear forms $L_1, L_2 \ldots, L_m$ is square independent if and only if it is controlled by Fourier analysis in the sense that for any $\epsilon > 0$, if $f : \mathbb{F}_p^n \to \mathbb{C}$ is any function satisfying $\|f\|_\infty \leq 1$ and $\|f\|_{U^2} \leq \epsilon$, then

$$|\mathbb{E}_{x_1,x_2,\ldots,x_d} \prod_{i=1}^{m} f(L_i(x_1, x_2, \ldots, x_m))| < c(\epsilon)$$

for some function $c(\epsilon)$ which tends to zero as ϵ tends to zero.

This suggests that *if* a square-independent configuration is uncommon, it suffices to look for bad colourings that exhibit a linear bias. To illustrate this phenomenon we give just one of numerous possible examples here.

Example 8.1 (A square-independent configuration)**.** Let $Z = \mathbb{F}_5^n$. Consider the configuration SI given by the system of equations

$$\begin{array}{rcl} x+y & = & 2u \\ x+y+z & = & 3v \end{array}.$$

It is square independent over $\mathbb{F}_5$, and the usual colouring of $\mathbb{F}_5^n$ consisting of ± 1 cosets of a subspace V of codimension 1 together with half of V yields

$$m_{\mathbb{F}_5^n, SI}(A, A^C) = 0.0525 < 2 \cdot \left(\frac{1}{2}\right)^5.$$

Conversely, uncommon square-dependent patterns such as 4-term arithmetic progressions require the construction of bad colourings with genuinely quadratic structure. Again, this is borne out in extensive empirical evidence which we shall not include here.

This alone does not tell us when a linear system is common or uncommon. As in the case of a single equation, we suspect that symmetry in the coefficients plays a major role, but we refrain from putting forward a precise conjecture to this effect.

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