MATH 592 SPRING 2010
TOPICS IN ERGODIC THEORY AND PROBABILITY

PROBLEM SET 1

1. Let $f : \mathbb{F}_p^n \to \mathbb{C}$, let V be a subspace of $\mathbb{F}_p^n$, and write $g = f \cdot 1_V$. Show that

$$\widehat{g}(t) = \mathbb{E}_{s \in V^\perp} \widehat{f}(s+t)$$

for all $t \in \mathbb{F}_p^n$, and deduce the *Poisson summation formula*

$$\mathbb{E}_{x \in V} f(x) = \mathbb{E}_{t \in V^\perp} \widehat{f}(t).$$

Conversely, show that if $h = f * \mu_V$, then $\widehat{h} = \widehat{f} \cdot 1_{V^\perp}$. (Here μ_V is the so-called *characteristic measure* of the subspace V, meaning its indicator function divided by its density.)

2. Choose a subset A of $\mathbb{F}_p^n$ randomly by letting $x \in A$ with probability p, where all these events are independent. Obtain an estimate for $\sup_{t \neq 0} |\widehat{1_A}(t)|$. (Hint: You may wish to compute the ℓ^4 norm of the balanced function of A.)

3. Let $A \subseteq \mathbb{Z}_N$ and suppose that $|A| \leq \frac{1}{10} \log N$. Prove that there exists a $t \neq 0$ such that $|\widehat{1}_A(t)| \geq \alpha/2$.

4. Let $A \subseteq \mathbb{F}_p^n$. Show that the number of 3-term progressions in A plus the number of 3-term progressions in A^C depends only on the cardinality of A. Is the same true for 4-term progressions?

5. Prove the following extension of Meshulam's Theorem: Let $a_1 + a_2 + a_3 = 0$. Then for every $\alpha > 0$, there exists n such that every $A \subseteq \mathbb{F}_p^n$ of cardinality at least αp^n contains elements x_1, x_2, x_3, not all equal, such that $a_1 x_1 + a_2 x_2 + a_3 x_3 = 0$.

6. Show that Meshulam's Theorem implies *van der Waerden's Theorem* for progressions of length 3 in $\mathbb{F}_p^n$. (In this context van der Waerden's Theorem states that if you k-colour the elements of $\mathbb{F}_p^n$, then provided n is large enough you are guaranteed a monochromatic 3-AP.)

Please send any comments or corrections to jwolf137@math.rutgers.edu.