MATH 592 SPRING 2010
TOPICS IN ERGODIC THEORY

PROBLEM SET 4

1. Show that T is ergodic if and only if the only T-invariant measurable functions are the functions which are constant almost everywhere.

2. Show that the map $D : \mathbb{T} \to \mathbb{T}$ which sends $x \mapsto 2x$ is ergodic.

3. Give an example to show that it is possible for T and T^{-1} to be ergodic, without T^n being ergodic for all n.

4. Prove the following extension of Poincaré's recurrence theorem. If $(X, \mathcal{X}, \mu, T)$ is a measure-preserving system, and $f \in L^1(X, \mathcal{X}, \mu, T)$ is non-negative, then

$$\limsup_{n\to\infty} \int_X f T^n f \geq \left(\int_X f \, d\mu \right)^2 .$$

5. Give an example to show that even in an ergodic system, the double average

$$\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x) f_2(T^{2n} x)$$

does not necessarily converge to a constant function.

6. To become familiar with Fourier anaysis on $\mathbb{T}$, prove at least two of the following statements:

(1) Define the Nth Fejér kernel $K_N : \mathbb{T} \to \mathbb{R}$ by $K_N(\theta) = \frac{1}{2N+1} \sum_{n=-N}^{N} e^{2\pi i \theta n}$. Show that $f * K_N \to f$ uniformly if $f \in C(\mathbb{T})$.

(2) Show that $f * K_N \to f$ in L^1 if $f \in L^1(\mathbb{T})$.

(3) For $f \in L^1(\mathbb{T})$, write $f * K_N$ in terms of the Fourier coefficients $\widehat{f}(t)$.

(4) Show that if $f, g \in L^1(\mathbb{T})$ and if $\widehat{f}(t) = \widehat{g}(t)$ for all $t \in \mathbb{Z}$, then $f = g$ almost everywhere.

(5) For $f \in C^\infty(\mathbb{T})$, show that $\sum_t \widehat{f}(t) e^{2\pi i \theta t}$ converges uniformly to f.

(6) For $f \in L^2(\mathbb{T})$, show that the partial sums $\sum_{|t| \leq N} \widehat{f}(t) e^{2\pi i \theta t}$ converge to f in L^2.

Please send any comments or corrections to jwolf137@math.rutgers.edu.