Further Topics in Analysis: Exercises 4

1. Let $X$ and $Y$ be sets.
   (a) Prove that $\mathcal{P}(X \cap Y) = \mathcal{P}(X) \cap \mathcal{P}(Y)$.
   (b) Give an example to show that $\mathcal{P}(X \cup Y)$ is not always the same as $\mathcal{P}(X) \cup \mathcal{P}(Y)$.
   (c) Give an example where $X \neq Y$ and $\mathcal{P}(X \cup Y) = \mathcal{P}(X) \cup \mathcal{P}(Y)$.
   (d) What condition must $X$ and $Y$ satisfy in order that $\mathcal{P}(X \cup Y) = \mathcal{P}(X) \cup \mathcal{P}(Y)$?

2. Let $\mathcal{A}$ be the set of all sequences of 0’s and 1’s:
   \[ \mathcal{A} = \{ (a_1, a_2, a_3, \ldots, a_k, \ldots) : a_k \in \{0, 1\} \}. \]
   (a) Use Cantor’s diagonalisation method to prove that $\mathcal{A}$ is uncountable. [Hint: Imitate the proof of Theorem 4.2.]
   (b) Deduce that the set $\mathcal{P}(\mathbb{N})$ is uncountable.

3. (a) Give an example of a sequence $(a_n)_{n \in \mathbb{N}}$ of real numbers that has a subsequence tending to 1 and a subsequence tending to 2.
   (b) Let $X = \{x_1, x_2, \ldots, x_s\}$ be a finite set of real numbers. Give an example of a sequence $(a_n)_{n \in \mathbb{N}}$ of real numbers that has subsequences tending to every element of $X$.
   (c) Give an example of a sequence $(a_n)_{n \in \mathbb{N}}$ of real numbers that has subsequences tending to every integer $m \in \mathbb{Z}$.

4. Find the sets of accumulation points of the following sequences:
   (a) $a_n = (-1)^n + \frac{1}{n}$;
   (b) $a_n = 1 + \frac{(-1)^n}{n}$;
   (c) $a_n = \frac{1}{n} + (-1)^n$;
   (d) $a_n = \frac{(-1)^n}{2n+1}$.

5. Let $\mathbb{Q}$ be the set of rational numbers. Since $\mathbb{Q}$ is countable we may enumerate the elements of $\mathbb{Q}$ as
   \[ \mathbb{Q} = \{ q_1, q_2, q_3, \ldots, q_n, \ldots \} \]
   (a) Construct a sequence that includes every rational number an infinite number of times.
   (b) Prove that, for every real number $x \in \mathbb{R}$, there is a subsequence of the sequence constructed in (a) that tends to $x$.

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