Further Topics in Analysis: Exercises 9

1. Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = x$. Prove that $f$ is Riemann integrable and compute
$$\int_0^1 f(x) \, dx$$
as the limit of upper (and lower) sums.

2. Prove that the function $f(x) = \sqrt{x}$ is Riemann integrable on $[0, 1]$ and compute
$$\int_0^1 \sqrt{x} \, dx$$
as the limit of upper (and lower) sums.

Hint: Consider the partition $Q_n$ of $[0, 1]$, given by
$$0 < \frac{1^2}{n^2} < \frac{2^2}{n^2} < \cdots < \frac{i^2}{n^2} < \cdots < \frac{(n-1)^2}{n^2} < 1$$
and compute the lower and the upper sums $L(f, P_n)$ and $U(f, P_n)$. Then compute the limit as $n \to \infty$.

3. (a) Let $g : [0, 1] \to \mathbb{R}$ be defined by
$$g(x) = \begin{cases} 
1 & \text{if } x = 1/2, \\
0 & \text{otherwise}.
\end{cases}$$

Show, using the Criterion of Integrability, that $g$ is integrable on $[0, 1]$.

(b) Let $h : [0, 1] \to \mathbb{R}$ be defined by
$$h(x) = \begin{cases} 
1 & \text{if } x = 1/n \text{ for some } n \in \mathbb{N}, \\
0 & \text{otherwise}.
\end{cases}$$

Show that $h$ is integrable on $[0, 1]$.

4. Construct a sequence $(g_n)_{n \in \mathbb{N}}$ of Riemann-integrable functions $g_n : [0, 1] \to \mathbb{R}$ converging pointwise to a function $g : [0, 1] \to \mathbb{R}$ which is not Riemann integrable.

Hint: You may wish to use the Criterion of Integrability to show that for each fixed $n \in \mathbb{N}$, $g_n$ is Riemann integrable on $[0, 1]$.

5. Let $f : [a, b] \to \mathbb{R}$ be a bounded function such that $f(x) = 0$ except for a finite number of points. Show that $f$ is Riemann integrable and that $\int_a^b f(x) \, dx = 0$. 

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