Further Topics in Analysis: Solutions 10

1. Show that if a sequence \((g_n)_{n \in \mathbb{N}}\) of Riemann-integrable functions \(g_n : [0, 1] \to \mathbb{R}\) converges uniformly to a function \(g : [0, 1] \to \mathbb{R}\), then \(g\) is Riemann integrable.

**Solution.** Since \(g_n \to g\) uniformly, given \(\varepsilon > 0\), there exists \(N \in \mathbb{N}\) (only depending on \(\varepsilon\)) such that for all \(n \geq N\) and all \(x \in [0, 1]\), \(|g_n(x) - g(x)| < \varepsilon/3\). Now let \(P = \{x_0, \ldots, x_k\}\) be any partition of \([0, 1]\). We observe that for all \(i = 1, \ldots, k\) and for all \(n \geq N\),

\[
\left| \sup_{x \in [x_{i-1}, x_i]} g_n(x) - \sup_{x \in [x_{i-1}, x_i]} g(x) \right| \leq \varepsilon/3
\]

and

\[
\left| \inf_{x \in [x_{i-1}, x_i]} g_n(x) - \inf_{x \in [x_{i-1}, x_i]} g(x) \right| \leq \varepsilon/3,
\]

so that for all \(n \geq N\),

\[
|U(g_n, P) - U(g, P)| = \left| \sum_{i=1}^{k} \sup_{x \in [x_{i-1}, x_i]} g_n(x)(x_i - x_{i-1}) - \sum_{i=1}^{k} \sup_{x \in [x_{i-1}, x_i]} g(x)(x_i - x_{i-1}) \right| < \varepsilon/3
\]

and similarly,

\[
|L(g_n, P) - L(g, P)| = \left| \sum_{i=1}^{k} \inf_{x \in [x_{i-1}, x_i]} g_n(x)(x_i - x_{i-1}) - \sum_{i=1}^{k} \inf_{x \in [x_{i-1}, x_i]} g(x)(x_i - x_{i-1}) \right| < \varepsilon/3.
\]

Now since \(g_n\) is integrable for each fixed \(n \in \mathbb{N}\), the Criterion of Integrability tells us that for the given \(\varepsilon > 0\) above, there exists a partition \(P_n^*\) such that

\[
U(g_n, P_n^*) - L(g_n, P_n^*) < \varepsilon/3.
\]

Putting everything together, we have that

\[
U(g, P_n^*) - L(g, P_n^*) \leq |U(g_n, P_n) - U(g, P)| + |U(g_n, P_n^*) - U(g_n, P_N^*)| + |U(g_N, P_n^*) - U(g, P_N^*)| + |L(g_n, P_n^*) - L(g, P_N)|< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.
\]

It follows from the Criterion of Integrability that \(g\) is integrable.

With only minor additions to the proof, you can go further and show that in fact,

\[
\lim_{n \to \infty} \int_0^1 g_n(x) \, dx = \int_0^1 \lim_{n \to \infty} g_n(x) \, dx = \int_0^1 g(x) \, dx
\]

whenever \(g_n \to g\) uniformly.

Even though a bit tricky, this is a very standard application which demonstrates the power of uniform convergence. You should compare this result with the example of a sequence of integrable functions converging pointwise to a non-integrable function on the previous sheet (Problem Sheet 9, Question 4).

2. Let \(f\) be a continuous real-valued function on \([a, b]\) such that

\[(\forall x \in [a, b])(f(x) \geq 0).
\]
Suppose that there exists \( x_0 \in [a, b] \) such that \( f(x_0) > 0 \). Prove that \[
\int_a^b f(x) \, dx > 0.
\]

**Solution.** Since \( f \) is continuous on \([a, b]\) it follows that

\[
(\exists \eta > 0)(\forall x \in [a, b]) \left[ |x - x_0| < \eta \Rightarrow (f(x) > \frac{1}{2}f(x_0)) \right].
\]

Suppose in the first instance that \( x_0 \in (a, b) \), i.e. \( x_0 \) is not one of the endpoints of the interval \([a, b]\). Let \( \delta = \min\{(x_0 - a)/2, (b - x_0)/2, \eta\} > 0 \). Then we can write

\[
\int_a^b f(x) \, dx = \int_a^{x_0-\delta} f(x) \, dx + \int_{x_0-\delta}^{x_0+\delta} f(x) \, dx + \int_{x_0+\delta}^b f(x) \, dx \\
\geq \int_{x_0-\delta}^{x_0+\delta} f(x) \, dx \\
> \frac{1}{2}f(x_0) \cdot 2\delta \\
> 0.
\]

If \( x_0 = b \) (and an almost identical argument applies if \( x_0 = a \)), then we can split the integral as

\[
\int_a^b f(x) \, dx = \int_a^{b-\eta} f(x) \, dx + \int_{b-\eta}^b f(x) \, dx \\
\geq \int_{b-\eta}^b f(x) \, dx \\
> \frac{1}{2}f(x_0) \cdot \eta \\
> 0.
\]

3. Let \( g \) be a continuous real-valued function on \([a, b]\).

(a) Prove that

\[
\int_a^b |g(x)| \, dx = 0
\]

if and only if \( g(x) = 0 \) for all \( x \in [a, b] \).

(b) Is the same statement true with the absolute-value signs removed? Give a proof or counterexample as appropriate.

(c) Is the same statement true when the continuity hypothesis is dropped? Give a proof or counterexample as appropriate.

**Solution.**

(a) If \( g(x) = 0 \) for all \( x \in [a, b] \), it is clear that

\[
\int_a^b |g(x)| \, dx = 0.
\]
For the reverse implication, suppose that
\[ \int_a^b |g(x)| \, dx = 0. \]
Define a function \( f : [a, b] \to \mathbb{R} \) by \( f(x) = |g(x)| \). Let us assume, for the sake of obtaining a contradiction, that there exists \( x \in [a, b] \) such that \( g(x) \neq 0 \). In that case \( f(x) > 0 \), so \( f \) satisfies the hypotheses of Q2, and by that result we have that
\[ \int_a^b |g(x)| \, dx = \int_a^b f(x) \, dx > 0, \]
contradicting our initial assumption above. It follows that \( g(x) = 0 \) for all \( x \in [a, b] \).

(b) When the absolute-value signs are removed this statement is clearly false. Take, for example, \( g(x) = x \) on \([-1, 1]\).

(c) On the previous sheet (Problem Sheet 9, Q3) we encountered the function \( g : [0, 1] \to \mathbb{R} \) defined by
\[ g(x) = \begin{cases} 1 & \text{if } x = 1/2, \\ 0 & \text{otherwise}, \end{cases} \]
and showed that it was integrable with integral 0. This function is non-negative, so
\[ \int_a^b |g(x)| \, dx = \int_a^b g(x) \, dx = 0, \]
but \( g(1/2) \neq 0 \), providing the desired counterexample.

4. Let \( f \) and \( g \) be a continuous real-valued functions on \([a, b]\), and suppose that
\[ (\forall x \in [a, b])(g(x) > 0). \]
Show that there exists \( c \in [a, b] \) such that
\[ \int_a^b f(x)g(x) \, dx = f(c) \int_a^b g(x) \, dx. \]
Give an example to show that this result may fail without the positivity hypothesis on \( g \).

**Solution.** Let \( m := \min_{x \in [a, b]} f(x) \) and \( M := \max_{x \in [a, b]} f(x) \). Note that these bounds exist and are attained since \( f \) is continuous on a closed bounded interval. Since \( g \) is positive everywhere on \([a, b]\), we have that
\[ mg(x) \leq f(x)g(x) \leq Mg(x) \]
for all \( x \in [a, b] \). By Theorem 16.1 therefore,
\[ m \int_a^b g(x) \, dx = \int_a^b mg(x) \, dx \leq \int_a^b f(x)g(x) \, dx \leq \int_a^b Mg(x) \, dx = M \int_a^b g(x) \, dx. \]

(Strictly speaking, we will only prove the first and last equalities in Theorem 17.4, but the proof is very straightforward and left as an exercise.) It follows from the above that
\[ m \leq \frac{\int_a^b f(x)g(x) \, dx}{\int_a^b g(x) \, dx} \leq M. \]
Since both \( m \) and \( M \) are values taken by \( f \) on the interval \([a, b]\), the Mean Value Theorem applied to \( f \) implies that there exists \( c \in [a, b] \) such that

\[
f(c) = \frac{\int_a^b f(x)g(x) \, dx}{\int_a^b g(x) \, dx},
\]

which is the desired result.

In order to see that this result may fail without the positivity hypothesis on \( g \), consider for example the functions \( f(x) = g(x) = x \) on the interval \([-1, 1]\): the left-hand side of the equation in the statement of the question evaluates to \(2/3\), while the right-hand side equals 0 regardless of the choice of \( c \).

5. Let \( f : [0, 1] \to \mathbb{R} \) be defined by \( f(x) = e^{-x^2} \).

(a) By bounding the function above and below on the interval \([0, 1]\), show that

\[
1 - 1/e \leq \int_0^1 e^{-x^2} \, dx \leq 1.
\]

(b) Use the inequalities in (a) to crudely estimate the integral to within an error of 0.184.

(c) By using Taylor’s theorem with the Lagrange form of the remainder, approximate the integral

\[
\int_0^1 e^{-x^2} \, dx
\]

to within an error of 0.005.

*Hint:* You will need to consider terms in \( x \) up to and including degree 6 in the expansion of \( e^{-x^2} \).

**Solution.**

(a) Observe that for all \( x \in [0, 1] \),

\[
e^{-x} \leq e^{-x^2} \leq 1,
\]

so by Theorem 16.1,

\[
1 - 1/e = \int_0^1 e^{-x} \, dx \leq \int_0^1 e^{-x^2} \, dx \leq \int_0^1 1 \, dx = 1.
\]

(b) It follows immediately from Part (a) that we can estimate \( \int_0^1 e^{-x^2} \, dx \) by the value \(1 - 1/(2e) \approx 0.816 \) (the mean of the two bounds), to within an error of at most \(1/(2e) < 0.184\).

(c) By Taylor’s theorem, we can write

\[
e^{-x^2} = 1 - x^2 + \frac{1}{2} x^4 - \frac{1}{6} x^6 + R_3,
\]

where the remainder satisfies \( 0 \leq R_3 \leq x^8/24 \) for \( x \in [0, 1] \). It follows that

\[
\int_0^1 e^{-x^2} \, dx = \int_0^1 1 - x^2 + \frac{1}{2} x^4 - \frac{1}{6} x^6 \, dx + \int_0^1 R_3 \, dx = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \int_0^1 R_3 \, dx.
\]
By Theorem 16.1, we can estimate the integral of the remainder as

\[ 0 \leq \int_{0}^{1} R_3 \, dx \leq \int_{0}^{1} \frac{x^8}{24} \, dx \leq \frac{1}{9 \cdot 24} = \frac{1}{216} < 0.005. \]

Thus

\[ \int_{0}^{1} e^{-x^2} \, dx \approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} = \frac{26}{35}, \]

to within an error of less than 0.005.