Further Topics in Analysis: Solutions 11

1. Let \( f : [0,1] \to \mathbb{R} \) be a continuous function satisfying
\[
\int_0^x f(t) \, dt = \int_x^1 f(t) \, dt \quad \text{for all } x \in [0,1].
\]
Show that \( f(x) = 0 \) for all \( x \in [0,1] \).

**SOLUTION.** Define \( F(x) = \int_0^x f(t) \, dt \) for every \( x \in [0,1] \). Since \( f \) is continuous, we can apply Theorem 18.5 to conclude that \( F \) is differentiable with derivative \( f \) on \((0,1)\).

On the other hand, by Theorem 17.1 and the hypothesis, we have the equality
\[
F(x) = \int_1^0 f(t) \, dt - F(x)
\]
for all \( x \in [0,1] \), from which we conclude that
\[
F(x) = \frac{1}{2} \int_0^1 f(t) \, dt
\]
for all \( x \in [0,1] \). But the right-hand side is a constant (independent of \( x \)), so \( F'(x) = 0 \) for all \( x \in (0,1) \), and hence \( f(x) = F'(x) = 0 \) for all \( x \in (0,1) \). By continuity, \( f(x) = 0 \) also at the points \( x = 0 \) and \( x = 1 \), and hence \( f(x) = 0 \) for all \( x \in [0,1] \) as required.

2. Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function, and define \( g : \mathbb{R} \to \mathbb{R} \) by
\[
g(x) := \int_{x-1}^{x+1} f(t) \, dt
\]
for each \( x \in \mathbb{R} \).
Show that \( g \) is differentiable on \( \mathbb{R} \), and find \( g'(x) \).

**SOLUTION.** Fix \( x \in \mathbb{R} \), and choose \( M \in \mathbb{R}^+ \) such that \(-M < x - 1\). Then we can write
\[
g(x) = \int_{-M}^{x+1} f(t) \, dt - \int_{-M}^{x-1} f(t) \, dt,
\]
where both integrals on the right-hand side are well defined, by Theorem 18.5 and the continuity of \( f \). Moreover, we can compute the derivative of \( g \) at \( x \) as
\[
g'(x) = f(x+1) - f(x-1)
\]
by applying Theorem 18.5 separately to each integral (using linearity of differentiation).

3. Consider the sign function \( s : [-1,1] \to \mathbb{R} \) defined by
\[
s(x) := \begin{cases} 
+1 & \text{if } 0 < x \leq 1, \\
-1 & \text{if } -1 \leq x < 0, \\
0 & \text{if } x = 0.
\end{cases}
\]
(a) Show that $s$ is Riemann integrable on $[-1, 1]$ and that $\int_{-1}^{1} s(x) \, dx = 0$.

(b) Let

$$S(x) := \int_{-1}^{x} s(t) \, dt.$$ 

Show that $S(x) = |x| - 1$ for all $x \in [-1, 1]$.

(c) Does $S'(0)$ exist? Discuss your conclusion with reference to Theorem 18.5.

**Solution.**

(a) Note that the function $s$ is monotone on $[-1, 1]$, and therefore Riemann integrable by Theorem 15.1. To see that its Riemann integral over the interval $[-1, 1]$ equals 0, take a partition $P = \{-1, -\varepsilon/4, \varepsilon/4, 1\}$. The lower sum of $f$ with respect to $P$ is

$$L(f, P) = -(1 - \varepsilon/4) + (1 - \varepsilon/4) - \varepsilon/2,$$

while the upper sum is

$$U(f, P) = -(1 - \varepsilon/4) + (1 - \varepsilon/4) + \varepsilon/2,$$

so each goes to 0 as $\varepsilon$ tends to 0 (and so does their difference, which is $\varepsilon$).

(b) Fix $x \in [-1, 1]$ and argue directly as above using upper and lower sums. For $-1 \leq x < 0$, use the partition $P = \{-1, x\}$, for $x = 0$ use $P = \{-1, -\varepsilon, 0\}$ and for $0 < x \leq 1$ use $P = \{-1, -\varepsilon/4, \varepsilon/4, x\}$ (where $\varepsilon > 0$ needs to be chosen small enough in each case for these partitions to be well defined). The result follows from a direct computation (drawing the graph of the function will help).

(c) We know from Analysis I that the function $x \mapsto |x|$ is not differentiable at 0 (make sure you know how to argue this rigorously!), and thus $S'(0)$ does not exist. Theorem 18.5, which guarantees the existence of the derivative everywhere on the interval, only applies to continuous integrands, and the function $s$ in this question is not continuous at 0.

4. Consider the function $h : [0, 1] \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} 
0 & \text{if } x \notin \mathbb{Q}, \\
1/q & \text{if } x \in \mathbb{Q} \text{ and } x = p/q \text{ in lowest terms,}
\end{cases}$$

introduced in Example 15.3. For each $x \in [0, 1]$, let

$$H(x) := \int_{0}^{x} h(t) \, dt.$$ 

(a) Show that the function $H$ is identically 0 on the interval $[0, 1]$.

(b) Show that $H'$ exists and equals 0 for all $x \in [0, 1]$.

(c) Conclude that $H$ is not a primitive of $h$.

**Solution.**

(a) Fix $x \in [0, 1]$, and adapt the proof given in Example 15.3 in the lecture notes to the interval $[0, x]$ (instead of $[0, 1]$—there is practically no difference but you should write it out anyway).
(b) Since $H(x) = 0$ for all $x \in [0, 1]$, we conclude that $H'$ exists and equals 0 for all $x \in [0, 1]$. (Note that we are not using any general property of $h$ here (such as continuity, which we don’t have), but rather the fact that we can compute the integral $\int_0^1 h(t) \, dt$ explicitly.)

(c) $H$ is not a primitive of $h$ since it follows from Part (b) that $H'(x) \neq h(x)$ for all $x \in \mathbb{Q} \cap [0, 1]$.

5. Let $f, g : [a, b] \to \mathbb{R}$ be Riemann-integrable functions.

(a) Let $t \in \mathbb{R}$. Show that
\[
\int_a^b (tf(x) + g(x))^2 \, dx \geq 0.
\]

(b) Use Part (a) to show that for $t > 0$,
\[
2 \left| \int_a^b f(x)g(x) \, dx \right| \leq t \int_a^b f(x)^2 \, dx + \frac{1}{t} \int_a^b g(x)^2 \, dx.
\]

(c) Show that if $\int_a^b f(x)^2 \, dx = 0$, then $\int_a^b f(x)g(x) \, dx = 0$.

(d) By substituting a judicious value of $t$ in Part (b) above, prove the Cauchy-Bunyakovsky-Schwarz Inequality, namely
\[
\left| \int_a^b f(x)g(x) \, dx \right|^2 \leq \left( \int_a^b f(x)^2 \, dx \right) \left( \int_a^b g(x)^2 \, dx \right).
\]

**Solution.**

(a) This follows immediately from Theorem 16.1, observing that the square integrand is non-negative for any value of $t \in \mathbb{R}$.

(b) Expanding out the square and using the linearity property of the integral (Theorems 17.3 and 17.4), we find that
\[
\int_a^b (tf(x) + g(x))^2 \, dx = t^2 \int_a^b f(x)^2 \, dx + 2t \int_a^b f(x)g(x) \, dx + \int_a^b g(x)^2 \, dx,
\]
which is non-negative by Part (a). Dividing through by $t > 0$ and rearranging gives
\[
-2t \int_a^b f(x)g(x) \, dx \leq t \int_a^b f(x)^2 \, dx + \frac{1}{t} \int_a^b g(x)^2 \, dx,
\]
from which the desired result follows by taking absolute values and applying the triangle inequality.

(c) It follows from Part (b) that if $\int_a^b f(x)^2 \, dx = 0$, then
\[
2 \left| \int_a^b f(x)g(x) \, dx \right| \leq \frac{1}{t} \int_a^b g(x)^2 \, dx.
\]
Letting $t$ tend to $\infty$, we find that
\[
\left| \int_a^b f(x)g(x) \, dx \right| \leq 0,
\]
but since the absolute value of a real number is always non-negative, this implies that
\[
\left| \int_a^b f(x)g(x) \, dx \right| = \int_a^b f(x)g(x) \, dx = 0.
\]

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(d) If \( \int_a^b f(x)^2 \, dx \neq 0 \), we may substitute
\[
t = \left| \int_a^b f(x) g(x) \, dx \right| / \int_a^b f(x)^2 \, dx
\]
in Part (b) above to obtain
\[
2 \left| \int_a^b f(x) g(x) \, dx \right| \leq \int_a^b f(x) g(x) \, dx \left( \int_a^b f(x)^2 \, dx / \int_a^b g(x)^2 \, dx \right),
\]
which upon rearranging yields the desired result.

Note that in the only case where we are not allowed to make this substitution we may invoke Part (c), establishing the Cauchy-Bunyakovsky-Schwarz Inequality in full generality.