1. Let \( X \) and \( Y \) be sets.
(a) Prove that \( \mathcal{P}(X \cap Y) = \mathcal{P}(X) \cap \mathcal{P}(Y) \).
(b) Give an example to show that \( \mathcal{P}(X \cup Y) \) is not always the same as \( \mathcal{P}(X) \cup \mathcal{P}(Y) \).
(c) Give an example where \( X \neq Y \) and \( \mathcal{P}(X \cup Y) = \mathcal{P}(X) \cup \mathcal{P}(Y) \).
(d) What condition must \( X \) and \( Y \) satisfy in order that \( \mathcal{P}(X \cup Y) = \mathcal{P}(X) \cup \mathcal{P}(Y) \)?

\textbf{Solution.} (a) Let \( A \) be a set. Then
\[
A \in \mathcal{P}(X \cap Y) \iff A \subseteq X \cap Y \\
\iff (A \subseteq X) \land (A \subseteq Y) \\
\iff (A \in \mathcal{P}(X)) \land (A \in \mathcal{P}(Y)) \\
\iff A \in (\mathcal{P}(X) \cap \mathcal{P}(Y)),
\]
so \( \mathcal{P}(X \cap Y) = \mathcal{P}(X) \cap \mathcal{P}(Y) \).
(b) Let \( X = \{0\} \) and \( Y = \{1\} \). Then
\[
\mathcal{P}(X) \cup \mathcal{P}(Y) = \{\emptyset, \{0\}, \{1\}\},
\]
but
\[
\mathcal{P}(X \cup Y) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}.
\]
(c) If \( X = Y \), then
\[
\mathcal{P}(X \cup Y) = \mathcal{P}(X) = \mathcal{P}(X) \cup \mathcal{P}(Y).
\]
(d) One must be a subset of the other. For example, if \( X \subseteq Y \), then \( X \cup Y = Y \) and \( \mathcal{P}(X) \subseteq \mathcal{P}(Y) \), so
\[
\mathcal{P}(X \cup Y) = \mathcal{P}(Y) = \mathcal{P}(X) \cup \mathcal{P}(Y).
\]
But if neither \( X \) nor \( Y \) is a subset of the other, then there are elements \( x \in X \setminus Y \) and \( y \in Y \setminus X \), so
\[
\{x, y\} \in \mathcal{P}(X \cup Y),
\]
but
\[
\{x, y\} \notin \mathcal{P}(X) \cup \mathcal{P}(Y).
\]

2. Let \( \mathcal{A} \) be the set of all sequences of 0th and 1th:
\[
\mathcal{A} = \{(a_1, a_2, a_3, \ldots, a_k, \ldots) : a_k \in \{0, 1\}\}.
\]
(a) Use Cantor’s diagonalisation method to prove that \( \mathcal{A} \) is uncountable. [Hint: Imitate the proof of Theorem 4.2.]
(b) Deduce that the set \( \mathcal{P}(\mathbb{N}) \) is uncountable.

\textbf{Solution.} (a) We prove this by contradiction. Assume that the set \( \mathcal{A} \) is countable. Then we can arrange all elements of \( \mathcal{A} \) in a “list”
\[
\mathcal{A} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \ldots, \alpha_k, \ldots\},
\]
where each element $\alpha_k$ is a sequence of 0th and 1th:

$$\alpha_k = (a_{k1}, a_{k2}, a_{k3}, a_{k4}, a_{k5}, \ldots, a_{kk}, \ldots).$$

Hence we can rewrite our "list" of elements in $A$ as

$$\alpha_1 = (a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, \ldots, a_{1k}, \ldots)$$
$$\alpha_2 = (a_{21}, a_{22}, a_{23}, a_{24}, a_{25}, \ldots, a_{2k}, \ldots)$$
$$\alpha_3 = (a_{31}, a_{32}, a_{33}, a_{34}, a_{35}, \ldots, a_{3k}, \ldots)$$
$$\alpha_4 = (a_{41}, a_{42}, a_{43}, a_{44}, a_{45}, \ldots, a_{4k}, \ldots)$$
$$\alpha_5 = (a_{51}, a_{52}, a_{53}, a_{54}, a_{55}, \ldots, a_{5k}, \ldots)$$
$$\ldots = 0, \ldots$$
$$\alpha_k = (a_{k1}, a_{k2}, a_{k3}, a_{k4}, a_{k5}, \ldots, a_{kk}, \ldots)$$

Define the sequence $\beta \in A$ by

$$\beta := (b_1 b_2 b_3 b_4 b_5 \ldots b_k \ldots)$$

where

$$b_k := \begin{cases} 1, & \text{if } a_{kk} = 0, \\ 0, & \text{if } a_{kk} = 1. \end{cases}$$

Then $\beta$ is different from $\alpha_k$ for any $k \in \mathbb{N}$. The reason is that, for each $k \in \mathbb{N}$ the member $a_{kk}$ of the sequence $\beta$ differs from that the member $a_{kk}$ of the sequence $\alpha_k$. We arrived to a contradiction. Hence $A$ is uncountable.

(b) Let $X \in P(\mathbb{N})$. Define $f : P(\mathbb{N}) \to A$ by

$$f(X) := (a_1, a_2, a_3, \ldots, a_k, \ldots), \quad a_k := \begin{cases} 1, & \text{if } k \in X, \\ 0, & \text{if } k \not\in X. \end{cases}$$

The inverse $f^{-1} : A \to P(\mathbb{N})$ is

$$f^{-1}((a_1, a_2, a_3, \ldots, a_k, \ldots)) := \{k \in \mathbb{N} | a_k = 1\}.$$

So $f$ is a bijection and hence $P(\mathbb{N})$ is uncountable.

3. (a) Give an example of a sequence $(a_n)_{n \in \mathbb{N}}$ of real numbers that has a subsequence tending to 1 and a subsequence tending to 2.

(b) Let $X = \{x_1, x_2, \ldots, x_s\}$ be a finite set of real numbers. Give an example of a sequence $(a_n)_{n \in \mathbb{N}}$ of real numbers that has subsequences tending to every element of $X$.

(c) Give an example of a sequence $(a_n)_{n \in \mathbb{N}}$ of real numbers that has subsequences tending to every integer $m \in \mathbb{Z}$.

**Solution.** (a) For example, the sequence

$$(a_n)_{n \in \mathbb{N}} = (\underbrace{1, 2, 1, 2, 1, 2, 1, 2, \ldots}_{\text{alternating}}).$$

Such sequence can be written by the formula

$$a_n = \begin{cases} 1, & \text{if } n \text{ is odd}, \\ 2, & \text{if } n \text{ is even}. \end{cases}$$

Then $a_{2n-1} \to 1$, and $a_{2n} \to 2$ as $n \to \infty$. 

2
(b) For example, the sequence

\[(a_n)_{n \in \mathbb{N}} = (a_1, a_2, \ldots, a_s, a_1, a_2, \ldots, a_s, a_1, a_2, \ldots, a_s, \ldots).\]

Such sequence can be written by the formula

\[a_n = \begin{cases} x_1 & \text{if } n \equiv 1 \pmod{s} \\ x_2 & \text{if } n \equiv 2 \pmod{s} \\ \vdots \\ x_s & \text{if } n \equiv 0 \pmod{s} \end{cases} \]

Then \(a_{sn+1} \to x_1, \ a_{sn+2} \to x_2, \ldots,\) and \(a_{sn} \to x_s\) as \(n \to \infty.\)

(c) For example, the sequence

\[(a_n)_{n \in \mathbb{N}} = (0, -1, 0, 1, -2, -1, 0, 1, 2, -3, -2, -1, 0, 1, 2, 3, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots)\]

contains every integer an infinite number of times (there are many other sequences with this property which would work just as well), so for any integer \(m \in \mathbb{Z}\) there is a subsequence

\[(m, m, m, \ldots)\]

which tends to \(m\) as \(n \to \infty.\)

4. Find the sets of all accumulation points of the following sequences:
   
   (a) \(a_n = (-1)^n + \frac{1}{n};\)
   
   (b) \(a_n = 1 + \frac{(-1)^n}{n};\)
   
   (c) \(a_n = \frac{1+(-1)^n}{n};\)
   
   (d) \(a_n = \frac{(-1)^n n}{2n+1}.\)

**Solution.**

(a) One can see immediately that \((a_n)_{n \in \mathbb{N}}\) contains two convergent subsequences \((a_{2k-1})_{k \in \mathbb{N}}\) and \((a_{2k})_{k \in \mathbb{N}},\) where

\[a_{2k-1} \to -1 \quad \text{and} \quad a_{2k} \to 1.\]

So \(-1\) and 1 are accumulation points of \((a_n)_{n \in \mathbb{N}}.\) It seems fairly obvious that \((a_n)_{n \in \mathbb{N}}\) has no other accumulation points, so the set of all accumulation points of \((a_n)_{n \in \mathbb{N}}\) is \([-1, 1].\)

One can prove rigorously that \((a_n)_{n \in \mathbb{N}}\) has no accumulation points different from \([-1, 1]\) as follows. Fix \(b \not\in \{-1, 1\}.\) To prove that \(b\) is not an accumulation point of \((a_n)_{n \in \mathbb{N}},\) we verify the negation of Prop. 7.12: A number \(a \in \mathbb{R}\) is not an accumulation point of a sequence \((a_n)_{n \in \mathbb{N}}\) if and only if there is an \(\varepsilon > 0\) such that the set \(\{n \in \mathbb{N} : a - \varepsilon < a_n < a + \varepsilon\}\) is finite. In other words, we need to verify that

\[\exists \varepsilon > 0 \forall N \in \mathbb{N} \exists n \in \mathbb{N} \geq N \Rightarrow |a_n - b| > \varepsilon.\]

Fix \(\varepsilon = \frac{1}{2} \min\{|1-b|, |1+b|\}.\) Then by the inequality \(|x - y| \geq ||x| - |y||\) we obtain

\[\left|\left((-1)^n + \frac{1}{n}\right) - b\right| \geq |(-1)^n - b| - \frac{1}{n} \geq \min\{|1-b|, |1+b|\} - \frac{1}{n} \geq 2\varepsilon - \frac{1}{n}.\]
Choose \( N \in \mathbb{N} \) such that \( \frac{1}{N} < \varepsilon \). Then

\[
(\forall n \in \mathbb{N}) \left( n \geq N \Rightarrow \left( \left| \left( -1 \right)^n + \frac{1}{n} - b \right| > \varepsilon \right) \right).
\]

This means that \( b \) is not an accumulation point of \( (a_n)_{n \in \mathbb{N}} \), that is the set of all accumulation points is indeed \( \{-1, 1\} \).

(b) One can easily see that \( \lim_{n \to \infty} a_n = 1 \). Then by Lemma 7.6 any subsequence of \( (a_n)_{n \in \mathbb{N}} \) also converges to 1. This means that the set of all accumulation points of \( (a_n)_{n \in \mathbb{N}} \) is \( \{1\} \).

(c) As above, one can easily see that \( \lim_{n \to \infty} a_n = 0 \). Then by Lemma 7.6 we conclude that any subsequence of \( (a_n)_{n \in \mathbb{N}} \) converges to 1. This means that the set of all accumulation points of \( (a_n)_{n \in \mathbb{N}} \) is \( \{0\} \).

(d) Rewrite the sequence \( (a_n)_{n \in \mathbb{N}} \) as

\[
a_n = \left( -1 \right)^n \frac{n}{2n + 1} = \left( -1 \right)^n \left( \frac{1}{2} - \frac{1}{4n + 2} \right).
\]

One can see that \( (a_n)_{n \in \mathbb{N}} \) contains two convergent subsequences \( (a_{2k-1})_{k \in \mathbb{N}} \) and \( (a_{2k})_{k \in \mathbb{N}} \), where

\[
a_{2k-1} \to -\frac{1}{2} \quad \text{and} \quad a_{2k} \to \frac{1}{2}.
\]

It is clear that \( (a_n)_{n \in \mathbb{N}} \) has no other accumulation points, so the set of all accumulation points of \( (a_n)_{n \in \mathbb{N}} \) is \( \{-1/2, 1/2\} \).

One can rigorously prove that \( (a_n)_{n \in \mathbb{N}} \) has no other accumulation points as follows. Fix \( b \notin \{-1/2, 1/2\} \). To prove that \( b \) is not an accumulation point of \( (a_n)_{n \in \mathbb{N}} \), we verify the negation of Prop. 7.12:

\[
(\exists \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N}) \left( (n \geq N) \Rightarrow (|a_n - b| > \varepsilon) \right).
\]

Fix \( \varepsilon = \frac{1}{2} \min\{|1/2 - b|, |1/2 - b|\} \). Then by the inequality \( |x - y| \geq ||x| - |y|| \) we obtain

\[
\left| \left( -1 \right)^n \left( \frac{1}{2} - \frac{1}{4n + 2} \right) - b \right| \geq \left| \left( -1 \right)^n \left( \frac{1}{2} - \frac{1}{4n + 2} \right) - b \right| \geq \min\left\{ \left| \frac{1}{2} - b \right|, \left| \frac{1}{2} - b \right| \right\} = \frac{1}{4n + 2} = 2\varepsilon - \frac{1}{4n + 2}.
\]

Choose \( N \in \mathbb{N} \) such that \( \frac{1}{4N+2} < \varepsilon \). Then

\[
(\forall n \in \mathbb{N}) \left( (n \geq N) \Rightarrow \left( \left| \left( -1 \right)^n \left( \frac{1}{2} - \frac{1}{4n + 2} \right) - b \right| > \varepsilon \right) \right).
\]

This means that \( b \) is not an accumulation point of \( (a_n)_{n \in \mathbb{N}} \), that is the set of all accumulation points is indeed \( \{-1/2, 1/2\} \).

5. Let \( \mathbb{Q} \) be the set of rational numbers. Since \( \mathbb{Q} \) is countable we can list elements of \( \mathbb{Q} \) in a sequence

\[
\mathbb{Q} = \{ q_1, q_2, q_3, \ldots, q_n, \ldots \}
\]

that includes every rational number exactly once.

(a) Construct a sequence (in terms of \( q_1, q_2, q_3, \ldots \)) that includes every rational number an infinite number of times.
(b) Prove that, for every real number \( x \in \mathbb{R} \), there is a subsequence of the sequence constructed in (a) that tends to \( x \).

**Solution.**  
(a) For example, the sequence

\[ (a_n)_{n \in \mathbb{N}} = (\sqrt{2}, q_1, q_2, \sqrt{2}, q_3, q_4, \sqrt{2}, q_5, q_6, \sqrt{2}, q_7, q_8, \ldots). \]

(b) Let \((a_n)_{n \in \mathbb{N}}\) be the sequence in (a). There is a sequence \((r_n)_{n \in \mathbb{N}}\) of rational numbers such that \(r_n \to x\) as \(n \to \infty\), and every sequence of rational numbers is a subsequence of \((a_n)_{n \in \mathbb{N}}\), because \(a_{n_1} = r_1\) for some \(n_1\). Since \(a_n = r_2\) for infinitely many values of \(n\), \(a_{n_2} = r_2\) for some \(n_2 > n_1\).

Since \(a_n = r_3\) for infinitely many values of \(n\), \(a_{n_3} = r_3\) for some \(n_3 > n_2\). And so on.