1. Let \( f : [0, 1] \to \mathbb{R} \) be defined by \( f(x) = x \). Prove that \( f \) is Riemann integrable and compute
\[
\int_0^1 f(x) \, dx
\]
as the limit of upper (and lower) sums.

**Solution.** Let \( P_n \) be the uniform partition of \([0, 1]\) given by
\[
0 < \frac{1}{n} < \frac{2}{n} \ldots < \frac{n-1}{n} < 1.
\]
The function \( f(x) = x \) is increasing, hence
\[
m_i = \inf_{[x_{i-1}, x_i]} f(x) = f(x_{i-1}) = \frac{i-1}{n} \quad \text{and} \quad M_i = \sup_{[x_{i-1}, x_i]} f(x) = f(x_i) = \frac{i}{n}.
\]
Using the formula
\[
\sum_{k=1}^{n} k = \frac{1}{2} n(n+1)
\]
we obtain
\[
L(f, P) = \sum_{i=1}^{n} \frac{i-1}{n} (x_i - x_{i-1}) = \sum_{i=1}^{n} \frac{i-1}{n} \frac{1}{n} = \frac{1}{n^2} \left( \frac{n(n+1)}{2} - n \right) = \frac{n-1}{2n},
\]
\[
U(f, P) = \sum_{i=1}^{n} \frac{i}{n} (x_i - x_{i-1}) = \sum_{i=1}^{n} \frac{i}{n} \frac{1}{n} = \frac{n+1}{2n}.
\]
Therefore
\[
\lim_{n \to \infty} L(f, P_n) = \frac{1}{2}, \quad \lim_{n \to \infty} U(f, P_n) = \frac{1}{2} \quad \text{and} \quad \lim_{n \to \infty} (U(f, P_n) - L(f, P_n)) = 0.
\]
By the Criterion of Integrability (Corollary 14.2) we conclude that the function \( f(x) = x \) is integrable on \([0, 1]\) and
\[
\int_0^1 x \, dx = \lim_{n \to \infty} L(f, P_n) = \lim_{n \to \infty} U(f, P_n) = \frac{1}{2}.
\]

2. Prove that the function \( f(x) = \sqrt{x} \) is Riemann integrable on \([0, 1]\) and compute
\[
\int_0^1 \sqrt{x} \, dx
\]
as the limit of upper (and lower) sums.

**Hint:** Consider the partition \( Q_n \) of \([0, 1]\), given by
\[
0 < \frac{1}{n^2} < \frac{2}{n^2} < \ldots < \frac{i^2}{n^2} < \ldots < \frac{(n-1)^2}{n^2} < 1
\]
and compute the lower and the upper sums $L(f, P_n)$ and $U(f, P_n)$. Then compute the limit as $n \to \infty$.

**SOLUTION.** It is convenient to use the partition $Q_n$ of $[0, 1]$, given by

$$0 < \frac{1^2}{n^2} < \frac{2^2}{n^2} < \cdots < \frac{i^2}{n^2} < \cdots < \frac{(n-1)^2}{n^2} < 1$$

(instead of the uniform partition). The function $f(x) = \sqrt{x}$ is increasing, hence

$$m_i = \inf_{[x_{i-1}, x_i]} f(x) = f(x_{i-1}) = \frac{i - 1}{n} \quad \text{and} \quad M_i = \sup_{[x_{i-1}, x_i]} f(x) = f(x_i) = \frac{i}{n}.$$  

Hence

$$L(f, Q_n) = \sum_{i=1}^{n} \frac{i - 1}{n} (x_i - x_{i-1}) = \sum_{i=1}^{n} \frac{i - 1}{n} \left( \frac{i^2}{n^2} - \frac{(i-1)^2}{n^2} \right)$$

$$= \frac{1}{n^3} \sum_{i=1}^{n} (i-1) \left( i^2 - (i-1)^2 \right) = \frac{1}{n^3} \left( \frac{2}{3} \sum_{i=1}^{n} i - \frac{2}{3} \sum_{i=1}^{n} i - 1 \right),$$

$$U(f, Q_n) = \sum_{i=1}^{n} \frac{i}{n} (x_i - x_{i-1}) = \sum_{i=1}^{n} \frac{i}{n} \left( \frac{i^2}{n^2} - \frac{(i-1)^2}{n^2} \right)$$

$$= \frac{1}{n^3} \sum_{i=1}^{n} i \left( i^2 - (i-1)^2 \right) = \frac{1}{n^3} \sum_{i=1}^{n} \left( 2i^2 - i \right).$$

Using the formulae

$$\sum_{k=1}^{n} 1 = n, \quad \sum_{k=1}^{n} k = \frac{1}{2} n(n+1), \quad \sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1),$$

we obtain that

$$L(f, Q_n) = \frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2}, \quad U(f, Q_n) = \frac{2}{3} + \frac{1}{2n} - \frac{1}{6n^2}.$$  

Therefore

$$\lim_{n \to \infty} L(f, Q_n) = \frac{2}{3}, \quad \lim_{n \to \infty} U(f, Q_n) = \frac{2}{3} \quad \text{and} \quad \lim_{n \to \infty} (U(f, Q_n) - L(f, Q_n)) = 0.$$  

By the Criterion of Integrability (Corollary 14.2) we conclude that the function $f(x) = \sqrt{x}$ is integrable on $[0, 1]$ and

$$\int_{0}^{1} \sqrt{x} \, dx = \lim_{n \to \infty} L(f, Q_n) = \lim_{n \to \infty} U(f, Q_n) = \frac{2}{3}.$$  

3. (a) Let $g : [0, 1] \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} 
1 & \text{if } x = 1/2, \\
0 & \text{otherwise}.
\end{cases}$$

Show, using the Criterion of Integrability, that $g$ is integrable on $[0, 1]$. 

(b) Let \( h : [0, 1] \rightarrow \mathbb{R} \) be defined by
\[
h(x) = \begin{cases} 
1 & \text{if } x = 1/n \text{ for some } n \in \mathbb{N}, \\
0 & \text{otherwise}.
\end{cases}
\]

Show that \( h \) is integrable on \([0, 1]\).

**Solution.** (a) Let \( \mathcal{P} = \{x_0, \ldots, x_n\} \) be a partition of \([0, 1]\) with \( x_{i-1} < 1/2 < x_i \). Then
\[
L(g, \mathcal{P}) = 0 \quad \text{and} \quad U(g, \mathcal{P}) = x_i - x_{i-1}.
\]

Hence
\[
U(g, \mathcal{P}) - L(g, \mathcal{P}) = x_i - x_{i-1}.
\]

Fix \( \varepsilon > 0 \). Choose a partition in such a way that \( x_i - x_{i-1} < \varepsilon \). By the Criterion of Integrability we conclude that \( g \) is integrable.

(b) Fix \( \varepsilon > 0 \). Construct a partition \( P \) as follows: let \( x_0 = 0, x_1 = \varepsilon/2 \), and let \( x_1, x_2, \ldots, x_m = 1 \) be a uniform partition of \([\varepsilon/2, 1]\) of norm \( \delta = \varepsilon^2/4 \). We need to estimate the difference between the upper and lower sums of \( h \) with respect to \( P \). The contribution to
\[
U(h, P) - L(h, P)
\]

from the interval \([0, \varepsilon/2]\) is, very crudely, at most \( \varepsilon/2 \). The contribution to \((*)\) from the interval \([\varepsilon/2, 1]\) is bounded above, again very crudely, by \( \delta \) times the number of points of the form \( 1/n \) in the interval \([\varepsilon/2, 1]\), which in turn is at most \( 2\varepsilon^{-1} \). It follows that \((*)\) is at most \( \varepsilon/2 + \delta \cdot 2\varepsilon^{-1} = \varepsilon \).

By the Criterion of Integrability we conclude that \( h \) is integrable.

4. Construct a sequence \((g_n)_{n \in \mathbb{N}}\) of Riemann-integrable functions \( g_n : [0, 1] \rightarrow \mathbb{R} \) converging pointwise to a function \( g : [0, 1] \rightarrow \mathbb{R} \) which is not Riemann integrable.

**Hint:** You may wish to use the Criterion of Integrability to show that for each fixed \( n \in \mathbb{N} \), \( g_n \) is Riemann integrable on \([0, 1]\).

**Solution.** Place any ordering on the rationals in \([0, 1]\) and list them (exhaustively) as a sequence \((a_n)_{n \in \mathbb{N}}\) (we can do this because the rationals are countable). For each \( n \in \mathbb{N} \), let \( g_n : [0, 1] \rightarrow \mathbb{R} \) be the function that takes the value \( g_n(x) = 1 \) if \( x \) equals one of the first \( n \) rationals on our list, and 0 otherwise. Clearly the pointwise limit of \( g_n \) as \( n \rightarrow \infty \) is the Dirichlet function \( g : [0, 1] \rightarrow \mathbb{R} \) where \( g(x) = 1 \) if \( x \in \mathbb{Q} \cap [0, 1] \) and \( g(x) = 0 \) otherwise. In Example 13.9 we showed that this function \( g \) is not Riemann integrable.

However, each individual function \( g_n : [0, 1] \rightarrow \mathbb{R} \) for fixed \( n \in \mathbb{N} \) is Riemann integrable. This follows from Q5 below, or from the following simpler argument. Given \( \varepsilon > 0 \), take a uniform partition of norm \( \delta > 0 \) such that \( \delta < \varepsilon/n \). The lower sum \( L(g_n, P) \) is clearly non-negative, and the upper sum \( U(g_n, P) \) is bounded above by \( \delta n < \varepsilon \). This means that
\[
U(g_n, P) - L(g_n, P) < \varepsilon - 0 = \varepsilon,
\]

so by the Criterion of Integrability \( g_n \) is Riemann integrable.

5. Let \( f : [a, b] \rightarrow \mathbb{R} \) be a bounded function such that \( f(x) = 0 \) except for a finite number of points in \((a, b)\). Show that \( f \) is Riemann integrable and that \( \int_a^b f(x) \, dx = 0 \).

**Solution.** Let the finite set of points be \( \{c_1, \ldots, c_k\} \), listed in increasing order. Since there are finitely many such points, the minimum distance between any two successive points is well defined
and strictly positive, so let \( c := \frac{1}{2} \min_{i=2,\ldots,k}(c_i - c_{i-1}) > 0 \). Write \( M \) and \( m \) for some upper and lower bound on \( f \) on \([a, b]\), respectively.

Fix \( \varepsilon > 0 \), and consider a partition which places a small interval of length \( \delta > 0 \) around each \( c_i \). More specifically, take \( \delta = \min\{\varepsilon/(2k(M - m)), c, 2(c_1 - a), 2(b - c_k)\} \), and write, for each \( i = 1,\ldots,k \), \( x_i^- = c_i - \delta/2 \) and \( x_i^+ = c_i + \delta/2 \). The conditions on \( \delta \) ensure that

\[
x_0^+ := a < x_1^- < x_1^+ < x_2^- < x_2^+ < \cdots < x_k^- < x_k^+ < x_{k+1}^- := b.
\]

Denote this partition by \( P \) (it is helpful to draw a picture of this set-up!). We can now compute

\[
L(f, P) = \sum_{i=1}^{k} \inf_{x \in [x_i^- , x_i^+]} f(x)(x_i^+ - x_i^-) + \sum_{i=1}^{k+1} \inf_{x \in [x_{i-1}^- , x_i^-]} f(x)(x_i^- - x_{i-1}^-) = \sum_{i=1}^{k} \inf_{x \in [x_i^- , x_i^+]} f(x)(x_i^+ - x_i^-),
\]

where the second sum disappears as by construction \( \inf_{x \in [x_{i-1}^- , x_i^-]} f(x) = 0 \) for each \( i = 1,\ldots,k+1 \). Similarly, we have

\[
U(f, P) = \sum_{i=1}^{k} \sup_{x \in [x_i^- , x_i^+]} f(x)(x_i^+ - x_i^-) + \sum_{i=1}^{k+1} \sup_{x \in [x_i^- , x_i^+]} f(x)(x_i^+ - x_i^-) = \sum_{i=1}^{k} \sup_{x \in [x_i^- , x_i^+]} f(x)(x_i^+ - x_i^-),
\]

and we find that

\[
U(f, P) - L(f, P) = \sum_{i=1}^{k} \left( \sup_{x \in [x_i^- , x_i^+]} f(x) - \inf_{x \in [x_i^- , x_i^+]} f(x) \right)(x_i^+ - x_i^-),
\]

which we can bound above by

\[
\sum_{i=1}^{k} (M - m)(x_i^+ - x_i^-) = k \cdot (M - m) \cdot \delta \leq \varepsilon.
\]

The conclusion that \( f \) is integrable and the value of the integral now follow from the Criterion of Integrability.