MITES 2010 ADVANCED CALCULUS
PROBLEM SET 1

DUE TUESDAY, JUNE 29

1. Find the arc length of the graph of $y = x^{3/2}$ over the interval $[1, 2]$.

2. Without your graphing calculator, draw the curve $r = e^{\theta}$ between $\theta = 0$ and $\theta = 2\pi$. Compute its arc length.

3. Sketch the polar curves C_1 given by $r = 4\cos\theta$ and C_2 given by $r = 1$. Find the area of the region enclosed by C_1 but outside C_2.

4. Find the cosine of the angle between the vectors (i) $\mathbf{a} = (2, 4, -3)$ and $\mathbf{b} = (1, 3, 2)$; (ii) $\mathbf{a} = (2, 3, -1)$ and $\mathbf{b} = (3, -5, 2)$.

5. Use vector algebra to prove that the angle inscribed in a semicircle must be a right angle. Hint: Label the radius vectors appropriately.

6. Show that $\mathbf{a} - proj_{\mathbf{b}}(\mathbf{a})$ and $\mathbf{b}$ are perpendicular, and interpret this fact geometrically.

7. Show that the area A of the triangle, two of whose sides are represented by the vectors $\mathbf{a}$ and $\mathbf{b}$, takes the form $\frac{1}{2}\|\mathbf{a} \times \mathbf{b}\|$.

8. Determine if the three vectors $(3, 2, -1)$, $(2, -1, 3)$ and $(1, -2, 2)$ are coplanar. In case your answer is negative, evaluate the volume of the parallelepiped spanned by these three vectors.

9. Show that $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$, and verify the formula in the case when $\mathbf{A} = (1, 3, 2)$, $\mathbf{B} = (2, 5, -1)$ and $\mathbf{C} = (1, 2, 3)$.

10. Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$ for $u = \frac{x+y+z}{\sqrt{x^2+y^2+z^2}}$.

11. The two sides forming the right angle of a right-angled triangle are denoted by a and b. The hypthenuse is h. If there are possible errors of ± 0.5 per cent in measuring a and b, find the maximum possible error in calculating (a) the area of the triangle and (b) the length of h.

12. The total surface area S of a cone of base radius r and perpendicular height h is given by $S = \pi r^2 + \pi r\sqrt{r^2 + h^2}$. If r and h are each increasing at the rate of $0.25m/s$, find the rate at which S is increasing at the instant when $r = 3m$ and $h = 4m$.

13. Let $f(x, y, z) = xyz$ and $y = e^{xz}$. Use the chain rule to find the value of $\frac{\partial f}{\partial z}$ when $x = 1$ and $z = 1$.

14. * Let S be the solid obtained by rotating the region below the graph of $y = \frac{1}{x}$ about the x-axis for $1 \leq x < \infty$.

(1) Compute the volume of S.

(2) Show that the surface area of S is infinite.

That is, if S were a container, you could fill its volume with a finite amount of paint, but you could not paint its surface with a finite amount of paint.

15. * Compute the arc length of the standard parabola $y = x^2$ between $x = 1$ and $x = 2$. This will require at least three advanced integration techniques applied in sequence, one of which we have not covered in this class.

16. * Let $c(t) = (x(t), y(t))$ be a parameterized curve such that $x'(t) > 0$ and $y(t) > 0$. Show that the area under $c(t)$ for $t_0 \leq t \leq t_1$ is

$$A = \int_{t_0}^{t_1} y(t)x'(t)\, dt.$$

Hint: $x(t)$ is strictly increasing and therefore has an inverse, say, $t = g(x)$.

17. * Show that for the circle $r = \sin\theta + \cos\theta$,

$$\frac{dy}{dx} = \frac{\cos(2\theta) + \sin(2\theta)}{\cos(2\theta) - \sin(2\theta)}.$$

Find the polar coordinates of the points at which the tangent line is horizontal.

Please send any comments or corrections to julia.wolf@cantab.net.