MITES 2010 ADVANCED CALCULUS
PROBLEM SET 2

DUE TUESDAY, JULY 6TH

1. To get started, let's compute some higher order partial derivatives:

(1) Calculate $\frac{\partial^4}{\partial z \partial z \partial w \partial x}\left[x^2 w^5 z^3 + \tan^{-1}(x^2 y z^3)\right]$.

(2) Show that $u(x,y) = \ln(x^2+y^2)$ is *harmonic*, i.e. it satisfies the *Laplace equation* $\Delta u = 0$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the *Laplace operator*.

2. The equation $F(x,y,z) = \cos(xy) - \sin(xz) = 0$ defines z as a function of x and y. Find $\frac{\partial z}{\partial x}$ as a function of x, y and z.

3. Find the directional derivative of f at P in the direction of $\mathbf{u}$ when (i) $f(x,y,z) = \log(x^2 + y^2 + z^2)$, $P = (0,0,1)$, $\mathbf{u} = (2,2,0) - P$; (ii) $f(x,y,z) = x\sin y + y \sin z + z \sin x$, $P = (\pi/2, 0, 0)$, $\mathbf{u} = (2\sqrt{3}, 2, 0)$; (iii) $f(x,y,z) = xye^z + yze^x$, $P = (1,0,0)$, $\mathbf{u} = (2,2,1) - P$.

4. Let $f(x,y,z)$ be a function defined throughout some region in three-dimensional space. Explain why the vector ∇f points in the direction in which f increases most rapidly, and why the length of the vector ∇f is the maximum rate of increase of f.

5. Show that for any two scalar fields A and B, $\nabla(AB) = A(\nabla B) + B(\nabla A)$. Verify this in the case where $A(x,y,z) = x^2yz + xz^2$ and $B(x,y,z) = xy^2z - z^3$.

6. Find the unit normal to the surface $4x^2y^2 - 3xz^2 - 2y^2z + 4 = 0$ at the point $(2,-1,-2)$, and hence determine an equation for the tangent plane at that point.

7. Suppose that the intersection of two surfaces $F(x,y,z) = 0$ and $G(x,y,z) = 0$ is a curve $\mathcal{C}$, and let P be a point on $\mathcal{C}$. Explain why $\nabla F \times \nabla G$ evaluated at P is a direction vector for the tangent line to $\mathcal{C}$ at P. Now let $F(x,y,z) = x^2 + y^2 + z^2 - 3$ and $G(x,y,z) = (x-2)^2 + (y-2)^2 + z^2 - 3$. Find a parametric representation of the tangent line to $\mathcal{C}$ at $P = (1,1,1)$.

8. Let $f(x,y) = x^2 + xy + y^2$, $x = v$ and $y = \sin(uv)$. Use the chain rule to find the value of $\frac{\partial f}{\partial v}$ when $u = \pi$ and $v = 1/2$.

9. The equation $F(x,y,z) = x^2yz + xyz^2 = 4$ defines z as a function of x and y. Find $\frac{\partial z}{\partial x}$ as a function of x, y and z.

10. Find the critical points of the functions

(1) $f(x,y) = x^2 + 2y^2 - 4y + 6x$
(2) $g(x,y) = x^2 - 12xy + y$

Use the second derivative test to determine the local minima, maxima and saddle points. Plot the surfaces using a computer to confirm your findings.

11. Determine the global extreme values of the function $f(x,y) = x^3 - 2y$ for $0 \leq x, y \leq 1$.

12. * The existence of partial derivatives is not a sufficient condition for differentiability, as the following example shows. Define $g(0,0) = 0$ and for $(x,y) \neq (0,0)$, let

$$g(x,y) = \frac{2xy(x+y)}{x^2+y^2}.$$

(1) Show that $g(x,y)$ is continuous at $(0,0)$.
(2) Show that both $\frac{\partial g}{\partial x}(0,0)$ and $\frac{\partial g}{\partial y}(0,0)$ exist and are equal to zero.
(3) Show that $g(x,y)$ is not locally linear at $(0,0)$ and hence not differentiable there. Hint: Consider $\lim_{h\to 0} \frac{g(h,h)}{h}$.
(4) Explain why $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ cannot both by continuous at $(0,0)$.
(5) Show that $\lim_{(x,y)\to(0,0)} \frac{\partial g}{\partial x}$ does not exist by examining the approach on the x- and the y-axis.

Please send any comments or corrections to julia.wolf@cantab.net.