MITES 2010 ADVANCED CALCULUS
PROBLEM SET 3

DUE TUESDAY, JULY 13TH

1. Find the maximum area of a rectangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

2. Find the point on the line of intersection of the planes $x + y + z = 1$ and $3x + 2y + z = 6$ that is closest to the origin.

3. Find the area bounded by the curve $x = y^2$ and $x = 2y - y^2$.

4. Show that the area outside the circle $r = a$ and inside the curve $r = 2a\cos\theta$ is given by $A = 2\int_0^{\pi/3}\int_a^{2a\cos\theta} r\,drd\theta$. Evaluate the integral.

5. By changing to polar coordinates, evaluate the integral $\int_0^\infty\int_0^\infty \frac{dx\,dy}{(1+x^2+y^2)^2}$.

6. Evaluate $I = \int_0^\infty e^{-x^2}\,dx$. You may wish to consider the expression $I^2 = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$.

7. Find the volume of the solid bounded by the planes $z = 0$, $y = 1$, $y = 3$, $x = 0$, $x = 3$ and the surface $z = x^2 + xy$.

8. Use cylindrical polar coordinates to find the volume of the region bounded above by the plane $z = 2x$ and below by the paraboloid $z = x^2 + y^2$.

9. Use a triple integral to find the volume of the region bounded by the sphere $x^2 + y^2 + z^2 = 4a^2$ and the cylinder $(x - a)^2 + y^2 = a^2$. It may help to draw a diagram in both the xy- and the xz-plane.

10. Evaluate $\int\int\int_S \sqrt{x^2 + y^2 + z^2}e^{x^2+y^2+z^2}dx\,dy\,dz$ over the sphere S of radius 2 which is centred at the origin.

11. * Given constants N, E, E_1, E_2 and E_3, consider the quantity

$$S(x_1, x_2, x_3) = x_1\ln(x_1) + x_2\ln(x_2) + x_3\ln(x_3)$$

subject to the constraints $x_1 + x_2 + x_3 = N$ and $E_1x_1 + E_2x_2 + E_3x_3 = E$. Show that there is a constant μ such that $x_i = C^{-1}e^{\mu E_i}$ for $i = 1, 2, 3$ with $C = (e^{\mu E_1} + e^{\mu E_2} + e^{\mu E_3})/N$.

If you interpret x_i as the number of molecules with given kinetic energy E_i, then you get the so-called *Boltzman distribution*. The quantity S is called *entropy*.

12. * We use double integration to evaluate the improper integral

$$J(a) = \int_0^\infty \frac{e^{-x} - e^{-ax}}{x}\, dx, \qquad a > 0.$$

(1) Show that $f(x) = \frac{e^{-x}-e^{-ax}}{x}$ can be made continuous at $x = 0$ by setting $f(0) = a - 1$.

(2) Prove that $J(a)$ converges.

(3) Establish the alternative representation $J(a) = \int_0^\infty \int_1^a e^{-xy}\, dy\, dx$.

(4) Prove that $J(a) = \ln a - \lim_{R\to\infty} \int_1^a \frac{e^{-Ry}}{y}\, dy$.

(5) Show that the limit in part (4) is zero, and hence $J(a) = \ln a$.

13. * Show that

$$\int\int_{\mathcal{D}} \ln\sqrt{x^2 + y^2}\, dA = -\frac{\pi}{2},$$

where $\mathcal{D}$ is the unit disk. Note that this is an improper integral (why?), so integrate first over the annulus $a \leq r \leq 1$.

14. * Use the map $x = \frac{\sin u}{\cos v}$ and $y = \frac{\sin v}{\cos u}$ to evaluate the integral

$$\int_0^1 \int_0^1 \frac{dx\, dy}{1 - x^2 - y^2}.$$

Again, this is an improper integral, but the change of variables shows that the result is finite.

Please send any comments or corrections to julia.wolf@cantab.net.