MITES 2010 ADVANCED CALCULUS
PROBLEM SET 4

DUE TUESDAY, JULY 20TH

1. Let $\mathcal{R}$ be the region contained between the lines $y = 4 - x$, $y = x + 1$ and $y = \frac{1}{3}(x - 4)$. Sketch the region $\mathcal{R}'$ obtained by setting $x = \frac{1}{2}(u + v)$ and $y = \frac{1}{2}(u - v)$. Use this change of variables to evaluate the integral $\int\int_{\mathcal{R}} xy\ dxdy$.

2. Evaluate the integral $\int\int_{\mathcal{R}} \frac{y}{x}$, where $\mathcal{R}$ is the region enclosed by the curves $x^2 - y^2 = 1$, $x^2 - y^2 = 4$, $y = 0$ and $y = x/2$. Hint: Set $u = x^2 - y^2$, $v = y/x$.

3. Set up but do *not* evaluate the integral $I = \int_{\mathbf{c}} xy\ ds$, where the 2-dimensional curve $\mathbf{c}$ is defined by the parametric equations $x = \cos^3 t$, $y = \sin^3 t$ from $t = 0$ to $t = \pi/2$. How would you proceed with the actual integration?

4. Let the vector field $\mathbf{F}$ be given by $\mathbf{F}(x, y, z) = (x^2, -2xy, yz)$, and suppose the 3-dimensional curve $\mathbf{c}$ has parametric representation $x = 2u$, $y = u^2$, $z = 3u - 1$. Evaluate $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{r}$ between the points $(2, 1, 2)$ and $(4, 4, 5)$.

5. Determine which of the following vector fields are conservative in $\mathbb{R}^3$, and find the corresponding potential function where appropriate: (i) $\mathbf{F}(x, y, z) = (2xy + z, x^2 + 2yz, x + y^2)$; (ii) $\mathbf{F}(x, y, z) = (yz + 2y, xz + 2x, xy + 3)$; (iii) $\mathbf{F}(x, y, z) = (y \cos x \cos z, \sin x \cos z, -y \sin x \sin z)$.

6. What is a necessary and sufficient condition for $dz = P\ dx + Q\ dy$ to be an exact differential? Determine whether or not the following are exact differentials: (i) $dz = (1 + 8xy)dx + 5x^2\ dy$; (ii) $dz = (15y^2 e^{3x} + 2xy^2)\ dx + (10ye^{3x} + x^2 y)\ dy$; (iii) $dz = (4y^3 \cos 4x + 3x^2 \cos 2y)\ dx + (3y^2 \sin 4x - 2x^3 \sin 2y)\ dy$.

7. Integrate the following exact differentials to obtain the function z: (i) $dz = (8e^{4x} + 2xy^2)dx + (4 \cos 4y + 2x^2 y)\ dy$; (ii) $dz = (3y^2 \cos 3x - 3 \sin 3x)\ dx + (2y \sin 3x + 4)\ dy$; (iii) $dz = 2(y + 1)e^{2x}\ dx + (e^{2x} - 2y)\ dy$.

8. Evaluate $I = \oint_c (2x+y)\ dx + (3x-2y)\ dy$ taken in an anti-clockwise manner around the triangle with vertices at $O=(0,0)$, $A=(1,0)$ and $B=(1,2)$ (i) by computing the line integral directly; (ii) by using Green's Theorem.

9. Evaluate the line integral $I = \oint_c xy\ dx + (2x-y)\ dy$ around the boundary of the region bounded by the curves $y=x^2$ and $x=y^2$ using Green's Theorem.

10. Evaluate $\int_S F\ dS$ when the scalar field $F(x,y,z) = xyz^2$ exists over the surface S defined by $x^2+y^2=9$, bounded by $z=0$ and $z=2$ in the first octant.

11. Suppose that the vector field $\mathbf{F}$ is given by $\mathbf{F}(x,y,z) = (x^2, -y, 2z)$, and S is the surface $2x+y+2z=2$ bounded by the planes $x=0$, $y=0$, $z=0$ in the first octant. Evaluate $\int_S \mathbf{F}\cdot d\mathbf{S}$.

12. Show that for any vector field $\mathbf{F}$ and any scalar field ϕ the divergence operator satisfies the product rule $\nabla\cdot(\phi\mathbf{F}) = (\nabla\phi)\cdot\mathbf{F} + \phi(\nabla\cdot\mathbf{F})$. Verify this explicitly in the case when $\mathbf{F}(x,y,z) = (3e^{xy}, e^{z^2}, e^{yz}-e^{xy})$ and $\phi(x,y,z) = xyz$.

13. Show that for any vector field $\mathbf{A}$, $div(curl\mathbf{A}) = 0$. Verify this explicitly in the case when $\mathbf{A}(x,y,z) = (xy^2, -2yz, xyz)$.

14. * Use a change of three variables to prove that the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is equal to abc times the volume of the unit sphere.

15. * How much energy does it take to carry a 10kg backpack from sea level to the top of the Mount Everest? You may assume that the force of gravity $\mathbf{F}$ is constant $-mg$ in the vertical direction, but you may *not* use your high school formula for potential energy. Use what you have learnt about vector fields to explain why we do not need to specify the path taken.

Please send any comments or corrections to julia.wolf@cantab.net.