MITES 2010 ADVANCED CALCULUS
REVIEW SHEET FOR FINAL

1. The deflection y at the centre of a circular plate suspended at the edge and uniformly loaded is given by $y = \frac{kwd^4}{t^3}$, where w is the total load, d is the diameter, t is the thickness of the plate and k is constant. Calculate the approximate percentage change in y if w is increased by 3 per cent, d is decreased by 2.5 per cent and t is decreased by 6 per cent.

2. The base radius of a cone r is decreasing at the rate of 0.1 cm/s while the perpendicular height h is increasing at the rate of 0.6 cm/s. Find the rate at which the volume V is changing when $r = 2$ cm and $h = 3$ cm.

3. Let $z = u^3 + v^2u + wu^2$, $u = x - y$, $v = x^2y$ and $w = xy + y^2$. Use the chain rule to (i) find expressions for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in terms of x and y; (ii) determine the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ when $(x, y) = (1, 2)$.

4. The equation $z^2v + zv^2 - u^3 = 0$ defines z implicitly as a function of u and v. (i) Find expressions for $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ in terms of z, u and v; (ii) determine the values of these derivatives when $u = \sqrt[3]{30}$ and $v = 1$.

5. A rectangle with sides parallel to the axes is inscribed in the region bounded by the x and y axes and the line $x + 2y = 2$. Find the maximum area of this rectangle using the method of Lagrange multipliers.

6. Sketch the curve $r = 4\sin^2\theta$ and calculate the area enclosed by the curve in the upper half plane.

7. Sketch the curve $r = \sin^3\frac{\theta}{3}$ and calculate its total length.

8. Evaluate the integral $\int_V x^2z^3 + 2xy^2z\, dV$ over the volume V of a cylinder of height 1 and base radius 2 centred at the origin of the xy-plane.

9. Find the mass of the hollow region bounded by the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 1$ if the density of the solid contained in the hollow is directly proportional to the distance from the origin. Hint: This means you should integrate the function $f(x,y,z) = \rho\sqrt{x^2+y^2+z^2}$ for a fixed constant ρ over the given region.

10. Evaluate $\int\int_{\mathcal{R}}(x+y)^2\sin^2(x-y)\ dx\ dy$, where the region $\mathcal{R}$ is the square with vertices $(0,1)$, $(1,2)$, $(2,1)$ and $(1,0)$. Hint: Sketch the region $\mathcal{R}$, and identify a change of variables that simplifies the boundaries.

11. Consider the function $f(x,y) = x^3+y^3-3xy+2$. (i) Find the critical points of f, and determine their nature. (ii) What are the global extrema of f over the domain $0 \le x, y \le 1$?

12. Consider the integral $I = \int_{\mathbf{c}}(3x^2y+xy^2)\ ds$. Evaluate I along the path $\mathbf{c}$ given by (i) the straight line $y=2x$ between $(0,0)$ and $(1,2)$; (ii) the parameterized circle $x = 4\cos u$, $y = 4\sin u$ between $(4,0)$ and $(0,4)$.

13. Consider the integral $I = \int_{\mathbf{c}}(6x^2+8xy^3)dx + (12x^2y^2+12y^3)\ dy$. Evaluate I between $(0,0)$ and $(2,6)$ along the path $\mathbf{c}$ given by (i) the straight line $y=3x$; (ii) the parabola $y = \frac{3}{2}x^2$. (iii) Show that I is in fact independent of the path, and integrate the differential directly between the points $(0,0)$ and $(2,6)$.

14. State Green's Theorem in the plane. Use it to evaluate the line integral $I = \oint_{\mathbf{c}}(xy+1)\ dx + 3y^2x\ dy$ around the boundary of the region enclosed by the curves $y=x^3$ and $y=4x$.

15. Compute the volume of the parallelepiped spanned by the vectors $\mathbf{A} = (1,3,-2)$, $\mathbf{B} = (3,-1,4)$ and $\mathbf{C} = (-1,3,2)$. Find a unit vector which is orthogonal to both $\mathbf{A}$ and $\mathbf{B}$.

16. Let the pressure field P in $\mathbb{R}^3$ be given by $P(x,y,z) = x^2yz^3+3yx(2z+3)$. (i) Determine the directional derivative of P in the direction of $\mathbf{u} = (3,6,-2)$ at the point $Q = (2,1,-1)$. (ii) Find the direction and magnitude of the maximum pressure decrease at the point Q. (iii) Determine an equation for the plane tangent to the surface of constant pressure $P(x,y,z) = 2$ at the point Q.

17. Show that the *curl* of a gradient is always zero. Verify this in the case when the scalar potential ϕ is given by $\phi(x,y,z) = ye^{-x} - x^2ze^y$.

18. If the vector field $\mathbf{F}$ is given by $\mathbf{F}(x,y,z) = (xz, 5xy, yz)$, evaluate $\int_c \mathbf{F}\cdot d\mathbf{s}$ along the curve $x = u+2$, $y = 3u^2$, $z = 4u$ between the points $(2,0,0)$ and $(3,3,4)$.

19. A scalar field $F(x,y,z) = x + y$ exists over a surface S defined by $x^2 + y^2 + z^2 = 25$, bounded by the planes $x = 0$, $y = 0$ and $z = 0$ in the first octant. Evaluate $\int_S F dS$.

20. Suppose that the vector field $\mathbf{F}$ is given by $\mathbf{F}(x,y,z) = (x + y, -2z, 2y)$, and S is the surface $x^2 + y^2 + z^2 = 16$ for $z \geq 0$ bounded by the plane $z = 0$. Evaluate $\int_S \mathbf{F} \cdot d\mathbf{S}$.

21. State the Divergence Theorem. Verify it when the surface $\mathcal{S}$ is given by the the hemisphere $x^2 + y^2 + z^2 = 25$ in the region $z \geq 0$ (together with its base) and the vector field $\mathbf{F}$ is given by $\mathbf{F}(x,y,z) = (x,y,z)$.

22. A surface $\mathcal{S}$ consists of the open hemisphere $x^2 + y^2 + z^2 = 4$ in the region $z \geq 0$, with normal pointing outward. Under the assumption that the vector field $\mathbf{F}(x,y,z) = (2y, -x, xz)$ exists over the surface and around its boundary, use Stokes's Theorem to compute the surface integral $\int \int_{\mathcal{S}} \nabla \times \mathbf{F} \cdot d\mathbf{S}$.

23. Write a *short* essay on each of the following topics: (i) Maxwell's Equations; (ii) conservative vector fields; (iii) local and global extrema.

Please send any comments or corrections to julia.wolf@cantab.net.