MITES 2010 ADVANCED CALCULUS
REVIEW SHEET FOR FINAL–SKETCH SOLUTIONS

1. The deflection y at the centre of a circular plate suspended at the edge and uniformly loaded is given by $y = \frac{kwd^4}{t^3}$, where w is the total load, d is the diameter, t is the thickness of the plate and k is constant. Calculate the approximate percentage change in y if w is increased by 3 per cent, d is decreased by 2.5 per cent and t is decreased by 6 per cent.

Solution 1. The basic formula for small changes is $\Delta y \approx \frac{\partial y}{\partial w}\Delta w + \frac{\partial d}{\partial d}\Delta d + \frac{\partial y}{\partial t}\Delta t$, since y depends on three variables w, d and t. We compute the partial derivatives $\frac{\partial y}{\partial w} = \frac{kd^4}{t^3}$, $\frac{\partial y}{\partial d} = \frac{4kwd^3}{t^3}$ and $\frac{\partial y}{\partial t} = -\frac{kwd^4}{3t^4}$, and substitute $\Delta w = 0.03w$, $\Delta d = -0.025d$ and $\Delta t = 0.04t$ to obtain $\Delta y \approx -0.05y$, that is, we have a percentage change of -5 per cent.

2. The base radius of a cone r is decreasing at the rate of 0.1 cm/s while the perpendicular height h is increasing at the rate of 0.6 cm/s. Find the rate at which the volume V is changing when $r = 2$ cm and $h = 3$ cm.

Solution 2. The volume of a cone of radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$. (If you don't know this, can you find this formula by integration?) By the chain rule, $\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$. We thus find the partial derivatives $\frac{\partial V}{\partial r} = \frac{2}{3}\pi rh$ and $\frac{\partial V}{\partial h} = \frac{1}{3}\pi r^2$, and substitute $r = 2, h = 3, \frac{dr}{dt} = -0.1$ and $\frac{dh}{dt} = 0.6$ to get $\frac{dV}{dt} = 0.4\pi$ (cm/s).

3. Let $z = u^3 + v^2 u + wu^2$, $u = x - y$, $v = x^2 y$ and $w = xy + y^2$. Use the chain rule to (i) find expressions for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in terms of x and y; (ii) determine the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ when $(x, y) = (1, 2)$.

Solution 3. The chain rule says that $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial x}$, and similarly for $\frac{\partial z}{\partial y}$. The individual partial derivatives are easy to evaluate, and we obtain $\frac{\partial z}{\partial x} = (3u^2 + v^2 + 2uw) \cdot 1 + 2vu \cdot 2xy + u^2 \cdot y$. All that is left to do for part (i) is to substitute $u = x - y$, $v = x^2 y$ and $w = xy + y^2$ to obtain an expression in x and y, only, and then insert $x = 1$ and $y = 2$ for part (ii).

4. The equation $z^2v + zv^2 - u^3 = 0$ defines z implicitly as a function of u and v. (i) Find expressions for $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ in terms of z, u and v; (ii) determine the values of these derivatives when $u = \sqrt[3]{30}$ and $v = 1$.

Solution 4. We can write $f(z,u,v) = z^2v + zv^2 - u^3 = 0$, and hence by the chain rule $0 = \frac{\partial f}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial u} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial u}$, where $\frac{\partial v}{\partial u} = 0$ since v does not depend on u, and $\frac{\partial u}{\partial u} = 1$. Therefore, $\frac{\partial z}{\partial u} = -\frac{\partial f}{\partial u}/\frac{\partial f}{\partial z} = \frac{3u^2}{2zv+v^2}$. Similarly for $\frac{\partial z}{\partial v}$, which completes part (i). For part (ii), observe that when $u = \sqrt[3]{30}$ and $v = 1$, then z can take two possible values, namely 5 and -6, which you have to treat separately.

5. A rectangle with sides parallel to the axes is inscribed in the region bounded by the x and y axes and the line $x + 2y = 2$. Find the maximum area of this rectangle using the method of Lagrange multipliers.

Solution 5. The area of the rectangle will be given by $f(x,y) = xy$ (draw a picture!). We therefore want to maximize $f(x,y)$ with respect to the constraint $g(x,y) = x + 2y - 2 = 0$, and write $\nabla f - \lambda \nabla g = 0$ for some parameter λ to be determined. The equations obtained this way are $y - \lambda = 0$ and $x - 2\lambda = 0$, which gives $y = x/2$. Substituting this into the constrain yields $x = 1$, and thus $y = 1/2$, so the maximal area is $1/2$.

6. Sketch the curve $r = 4\sin^2\theta$ and calculate the area enclosed by the curve in the upper half plane.

Solution 6. The curve looks like an upright figure of eight, with its highest point at $(0,4)$ and its lowest point at $(0,-4)$. By the standard formula for the area of a polar curve, we have $A = \int_0^\pi (4\sin^2\theta)^2 \, d\theta = 8\int_0^\pi \sin^4\theta \, d\theta$. Using the double-angle formula to replace $\sin^2\theta$, we obtain $A = 2\int_0^\pi 1 - 2\cos 2\theta + \cos^2 2\theta \, d\theta$, and replacing $\cos^2 2\theta$ by the corresponding double angle formula gives $A = \int_0^\pi 3 - 4\cos\theta + \cos 4\theta \, d\theta = 3\pi$.

7. Sketch the curve $r = \sin^3\frac{\theta}{3}$ and calculate its total length.

Solution 7. Plot values $r = \sin^3\frac{\theta}{3}$ for θ between 0 and 3π to obtain Figure 1 below. In order to compute the total arc length, find $f'(\theta) = \sin^2(\frac{\theta}{3})\cos(\frac{\theta}{3})$, and thus $f(\theta)^2 + f'(\theta)^2 = \sin^4(\frac{\theta}{3})$ after some manipulations. The integral for the arc length is therefore $L = \int_0^{3\pi}\sqrt{\sin^4(\frac{\theta}{3})}\, d\theta = \int_0^{3\pi}\sin^2(\frac{\theta}{3})\, d\theta = \frac{3\pi}{2}$.

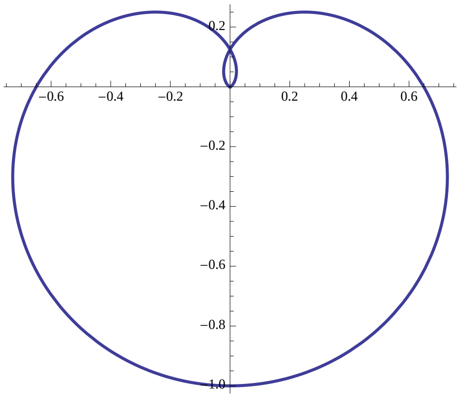

FIGURE 1. $f(\theta) = \sin^3(\frac{\theta}{3})$

8. Evaluate the integral $\int_V x^2z^3 + 2xy^2z \, dV$ over the volume V of a cylinder of height 1 and base radius 2 centred at the origin of the xy-plane.

Solution 8. This integral is most easily evaluated in cylindrical polars: set $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, use the volume element $r \, dr d\theta dz$ and the limits of integration $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq 1$. The resulting integral is $\int_0^1 \int_0^{2\pi} \int_0^2 (r^2\cos^2\theta z^3 + 2r\cos\theta r^2 \sin^2\theta z) r \, dr d\theta dz$. The only integral that is vaguely problematic is that in θ: for the first part, use the double-angle formula, for the second part, the substitution $u = \sin\theta$. The answer is π.

9. Find the mass of the hollow region bounded by the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 1$ if the density of the solid contained in the hollow is directly proportional to the distance from the origin. Hint: This means you should integrate the function $f(x, y, z) = \rho\sqrt{x^2 + y^2 + z^2}$ for a fixed constant ρ over the given region.

Solution 9. The total mass is $M = \int_{\mathcal{S}} \rho\sqrt{x^2 + y^2 + z^2} \, dxdydz$, where $\mathcal{S}$ is the region between the two spheres centered at the origin, with radius 1 and 2, respectively. This is an integral that is most easily computed by changing to spherical polar coordinates. Indeed, we have that $M = \int_0^\pi \int_0^{2\pi} \int_1^2 \rho r(r^2 \sin\phi) \, dr d\theta d\phi$. Computing the easy integral in θ first, we obtain $M = 2\pi\rho \int_0^\pi \int_1^2 r^3 \sin\phi \, dr d\phi$, which evaluates to $15\pi\rho$.

10. Evaluate $\int\int_{\mathcal{R}} (x + y)^2 \sin^2(x - y) \, dx \, dy$, where the region $\mathcal{R}$ is the square with vertices $(0, 1)$, $(1, 2)$, $(2, 1)$ and $(1, 0)$. Hint: Sketch the region $\mathcal{R}$, and identify a change of variables that simplifies the boundaries.

Solution 10. The region has diamond shape and lies in the first quadrant. The boundaries are $x+y=1$, $x+y=3$, $y-x=-1$ and $y-x=1$, and thus a natural change of variables would be $u=y+x$, $v=y-x$. Equivalently, we have $y=\frac{1}{2}(u+v)$ and $x=\frac{1}{2}(u-v)$. Thus the Jacobian $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ equals 1/2, and the integral becomes $\frac{1}{2}\int_{-1}^{1}\int_{1}^{3}u^2\sin^2 v\,dudv$, which evaluates to $\frac{13}{6}(2-\sin 2)$.

11. Consider the function $f(x,y)=x^3+y^3-3xy+2$. (i) Find the critical points of f, and determine their nature. (ii) What are the global extrema of f over the domain $0\leq x,y\leq 1$?

Solution 11. In order to find local extrema, examine the equations $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$. We obtain that $y=x^2$ and $x=y^2$, and hence the two points (0,0) and (1,1). Next we examine the discriminant $D=\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}-\frac{\partial^2 f}{\partial x\partial y}^2=36xy-9$ for these two points: At (0,0), $D<0$ and therefore (0,0) is a saddle point. At (1,1), $D>0$ and $\frac{\partial^2 f}{\partial x^2}>0$, thus (1,1) is a local minimum. For part (ii), we need to examine the boundaries of the square given. It is easy to see that on the bottom and left-hand boundary we have a min of 2 and a max of 3. On the top and right-hand boundary we must use one-variable calculus to determine the min and max. We find a max of 3 and a min of 1 at the end points. (Note that because of symmetry of the function f in the variables x and y, it is enough to consider one of these two boundaries). The conclusion is that (1,1) is a global minimum, and the points (0,1) and (1,0) are global maxima.

12. Consider the integral $I=\int_{\mathbf{c}}(3x^2y+xy^2)\;ds$. Evaluate I along the path $\mathbf{c}$ given by (i) the straight line $y=2x$ between $(0,0)$ and $(1,2)$; (ii) the parameterized circle $x=4\cos u$, $y=4\sin u$ between $(4,0)$ and $(0,4)$.

Solution 12. For part (i), you can think of $y=2x$ as the parameterized curve $\mathbf{c}(t)=(t,2t)$, so that $\mathbf{c}'(t)=(1,2)$ and $\|\mathbf{c}'(t)\|=\sqrt{5}$. The integral that needs to be computed is therefore $\int_0^1(3t^2(2t)+t(2t)^2)\sqrt{5}\;dt$. In part (ii), the parameterization is given as $\mathbf{c}(u)=(4\cos u,4\sin u)$, and therefore $\|\mathbf{c}'(u)\|=4$. The integral turns into $4^4\int_0^{\pi/2}3\cos^2 u\sin u+\cos u\sin^2 u\;du$. What's the integration technique needed to compute this integral? We perform a substitution (say $w=\cos u$ for the first part, and $w=\sin u$ for the second).

13. Consider the integral $I=\int_{\mathbf{c}}(6x^2+8xy^3)dx+(12x^2y^2+12y^3)\;dy$. Evaluate I between $(0,0)$ and $(2,6)$ along the path $\mathbf{c}$ given by (i) the straight line $y=3x$; (ii) the parabola $y=\frac{3}{2}x^2$. (iii)

Show that I is in fact independent of the path, and integrate the differential directly between the points $(0,0)$ and $(2,6)$.

Solution 13. For part (i), $I = \int_0^2 (6x^2 + 8x(3x)^3)dx + (12x^2(3x)^2 + 12(3x)^3)\ 3dx$ since $dy = 3\ dx$ when $y = 3x$. For part (ii), $I = \int_0^2 (6x^2 + 8x(\frac{3}{2}x^2)^3)dx + (12x^2(\frac{3}{2}x^2)^2 + 12(\frac{3}{2}x^2)^3)\ 3x\ dx$ since $dy = 3x\ dx$ when $y = \frac{3}{2}x^2$. (iii) I is independent of the path since $dz = (6x^2 + 8xy^3)dx + (12x^2y^2 + 12y^3)\ dy$ is an exact differential: check that $\frac{\partial}{\partial y}(6x^2 + 8xy^3) = \frac{\partial}{\partial x}(12x^2y^2 + 12y^3)$. Integrate dz by setting $\frac{\partial z}{\partial x}6x^2 + 8xy^3$ and $\frac{\partial z}{\partial y} = 12x^2y^2 + 12y^3$, so that $z = 2x^3 + 4x^2y^3 + 3y^4 + C$. Evaluating between the points $(1,3)$ and $(2,4)$, this gives 7360 (same result as in parts (i) and (ii)).

14. State Green's Theorem in the plane. Use it to evaluate the line integral $I = \oint_{\mathbf{c}}(xy+1)\ dx + 3y^2x\ dy$ around the boundary of the region enclosed by the curves $y = x^3$ and $y = 4x$.

Solution 14. If $\partial\mathcal{D}$ is a simple closed curve enclosing a domain $\mathcal{D}$ in the plane, and the derivatives of $P(x,y), Q(x,y)$ exist and are continuous on $\mathcal{D}$, then $\int\int_{\mathcal{D}} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dx\ dy = \oint_{\partial\mathcal{D}} P\ dx + Q\ dy$. In this example $P = xy + 1$ and $Q = 3y^2x$, both of which clearly satisfy the hypotheses of the theorem. So does the given region, which you should sketch. Green's Theorem tells us that $I = \int\int_{\mathcal{D}} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dxdy = \int\int_{\mathcal{D}}(3y^2 - x)\ dxdy = \int_0^2\int_{x^3}^{4x}(3y^2 - x)\ dydx$. The integral evaluates to $\frac{448}{3}$. (If you like you can check that the line integral gives the same value, but this is *not* required in this question!)

15. Compute the volume of the parallelepiped spanned by the vectors $\mathbf{A} = (1,3,-2)$, $\mathbf{B} = (3,-1,4)$ and $\mathbf{C} = (-1,3,2)$. Find a unit vector which is orthogonal to both $\mathbf{A}$ and $\mathbf{B}$.

Solution 15. The volume of the parallelepiped is given by $|\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})| = |(1,3,-2)\cdot(-14,-10,8)| = 60$. The unit vector orthogonal to both $\mathbf{A}$ and $\mathbf{B}$ is given by $\mathbf{A}\times\mathbf{B}/\|\mathbf{A}\times\mathbf{B}\| = 10(1,-1,-1)/10\sqrt{3} = (1,-1,-1)/\sqrt{3}$. (In fact, you can save yourself some work by computing the volume in the first part as $|\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})|$!)

16. Let the pressure field P in $\mathbb{R}^3$ be given by $P(x,y,z) = x^2yz^3 + 3yx(2z+3)$. (i) Determine the directional derivative of P in the direction of $\mathbf{u} = (3,6,-2)$ at the point $Q = (2,1,-1)$. (ii) Find the direction and magnitude of the maximum pressure decrease at the point Q. (iii) Determine an equation for the plane tangent to the surface of constant pressure $P(x,y,z) = 2$ at the point Q.

Solution 16. The gradient of P is $\nabla P = (2xyz^3 + 3y(2z+3), x^2z^3 + 3x(2z+3), 3x^2yz^2 + 6yx)$, which equals $(-1, 2, 24)$ at the given point Q. For part (i), we need to calculate the directional derivative $D_{\hat{\mathbf{u}}}P(Q) = \nabla P_Q \cdot \hat{\mathbf{u}}$, where $\hat{\mathbf{u}}$ is the normalized vector $\mathbf{u}$, i.e. $\hat{\mathbf{u}} = \frac{1}{7}(3, 6, -2)$. Taking the dot product, we find, $D_{\hat{\mathbf{u}}}P(Q) = -\frac{39}{7}$. For part (ii), the direction of the maximum pressure decrease is given by $-\nabla P_Q$, and its magnitude is $\|\nabla P_Q\| = \sqrt{581}$. Finally, the equation of the tangent plane required in part (iii) is $(\mathbf{x} - (2, 1, -1)) \cdot \nabla P_Q = 0$ since the gradient is perpendicular to lines of constant pressure, and hence to the surface $P(x, y, z) = 2$.

17. Show that the *curl* of a gradient is always zero. Verify this in the case when the scalar potential ϕ is given by $\phi(x, y, z) = ye^{-x} - x^2ze^y$.

Solution 17. The curl of the grad of a scalar potential ϕ is given by $\nabla \times \nabla\phi = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \times (\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}) = (\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}, \frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial^2\phi}{\partial x\partial z}, \frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}) = (0, 0, 0)$ since the mixed partial derivatives are equal. In this particular example, $\nabla\phi = (-ye^{-x} - 2xze^y, e^{-x} - x^2ze^y, -x^2e^y)$, the curl of which is zero.

18. If the vector field $\mathbf{F}$ is given by $\mathbf{F}(x, y, z) = (xz, 5xy, yz)$, evaluate $\int_c \mathbf{F} \cdot d\mathbf{s}$ along the curve $x = u + 2$, $y = 3u^2$, $z = 4u$ between the points $(2, 0, 0)$ and $(3, 3, 4)$.

Solution 18. This is a line integral over a vector field, so we know that $d\mathbf{s} = \mathbf{c}'(u)du$, where $\mathbf{c}(u) = (u + 2, 3u^2, 4u)$ and thus $\mathbf{c}'(u) = (1, 6u, 4)$. The integral we need to compute is therefore equal to $\int_0^1((u+2)(4u), 5(u+2)(3u^2), (3u^2)(4u)) \cdot (1, 6u, 4)\, du = \frac{241}{3}$.

19. A scalar field $F(x, y, z) = x + y$ exists over a surface S defined by $x^2 + y^2 + z^2 = 25$, bounded by the planes $x = 0$, $y = 0$ and $z = 0$ in the first octant. Evaluate $\int_S F dS$.

Solution 19. The surface is clearly a sphere, so we know from our experience with spherical polars what its parametrization ought to be: $\Phi(\theta, \phi) = (5\cos\theta\sin\phi, 5\sin\theta\sin\phi, 5\cos\phi)$, where $0 \le \theta, \phi \le \pi/2$. In order to determine the normal, compute $\frac{\partial\Phi}{\partial\theta} = (-5\sin\theta\sin\phi, 5\cos\theta\sin\phi, 0)$ and $\frac{\partial\Phi}{\partial\phi} = (5\cos\theta\cos\phi, 5\sin\theta\cos\phi, -5\sin\phi)$. Now the normal to the surface is $\mathbf{n} = \frac{\partial\Phi}{\partial\theta} \times \frac{\partial\Phi}{\partial\phi} = (-25\cos\theta\sin^2\phi, -25\sin\theta\sin^2\phi, -25\sin\phi\cos\phi) = -25\sin\phi(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$ (note how this makes sense, the normal vector ought to be in the radial direction!), and thus $\|\mathbf{n}\| = 25|\sin\phi|$ (again, this makes sense: it corresponds to the volume element $r^2\sin\phi$ which we use when changing to spherical polars!). Since $dS = \|\mathbf{n}\|\ d\theta d\phi$, our surface integral can be written as $\int_0^{\pi/2}\int_0^{\pi/2}(5\cos\theta\sin\phi + 5\sin\theta\sin\phi)25\sin\phi\ d\theta d\phi = \frac{125}{2}\pi$.

20. Suppose that the vector field $\mathbf{F}$ is given by $\mathbf{F}(x,y,z) = (x+y, -2z, 2y)$, and S is the surface $x^2+y^2+z^2 = 16$ for $z \geq 0$ bounded by the plane $z = 0$. Evaluate $\int_S \mathbf{F}\cdot d\mathbf{S}$.

Solution 20. The surface is a hemisphere of radius 4. Again, it seems advisable to parameterize as in the preceding question: $\Phi(\theta,\phi) = (4\cos\theta\sin\phi, 4\sin\theta\sin\phi, 4\cos\phi)$, where $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi/2$. Proceeding exactly as above, we find that the normal is $\mathbf{n} = -16\sin\phi(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$, and this time $d\mathbf{S} = \mathbf{n}\ d\theta d\phi$ so that the surface integral over the given vector field becomes $\int_0^{2\pi}\int_0^{\pi/2} 16\sin\phi(4\cos\theta\sin\phi + 4\sin\theta\sin\phi, -8\cos\phi, 8\sin\theta\sin\phi)\cdot(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)\ d\phi d\theta = \int_0^{2\pi}\int_0^{\pi/2} 64\sin^3\phi\cos\theta(\cos\theta + \sin\theta)\ d\phi d\theta = \frac{128}{3}\pi$.

21. State the Divergence Theorem. Verify it when the surface $\mathcal{S}$ is given by the the hemisphere $x^2+y^2+z^2 = 25$ in the region $z \geq 0$ (together with its base) and the vector field $\mathbf{F}$ is given by $\mathbf{F}(x,y,z) = (x,y,z)$.

Solution 21. The Divergence Theorem states that, if $\mathcal{S}$ is piecewise smooth surface enclosing a volume V, with outward normal, then $\int\int\int_{\mathcal{V}} \nabla\cdot\mathbf{F}\ dV = \int\int_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S}$. We thus need to compute a volume and a surface integral to verify the theorem. The volume integral of the divergence $\nabla\cdot\mathbf{F} = 1+1+1 = 3$ is easy since the divergence is constant, so we can just multiply the divergence by the volume of a sphere of radius 5 to obtain 500π. (Or else we integrate in spherical polars $\int_0^{2\pi}\int_0^{\pi}\int_0^5 3r^2\sin\phi\ dr d\phi d\theta$, to the same effect.) Now the surface integral: by repeating the steps in the preceding question, we find that the normal is $\mathbf{n} = 25\sin\phi(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$ (where this time we have removed the minus sign, since orientation matters!). Here $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$ since we are considering the entire sphere. Thus the surface integral becomes $\int_0^{2\pi}\int_0^{\pi} 25\sin\phi(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)\cdot(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)\ d\phi d\theta$, which after the appropriate simplifications equals $\int_0^{2\pi}\int_0^{\pi} 125\sin\phi\ d\phi d\theta = 500\pi$ as before.

22. A surface $\mathcal{S}$ consists of the open hemisphere $x^2+y^2+z^2 = 4$ in the region $z \geq 0$, with normal pointing outward. Under the assumption that the vector field $\mathbf{F}(x,y,z) = (2y, -x, xz)$ exists over the surface and around its boundary, use Stokes's Theorem to compute the surface integral $\int\int_{\mathcal{S}} \nabla\times\mathbf{F}\cdot d\mathbf{S}$.

Solution 22. By Stokes's Theorem, we need to compute the line integral $\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}$ around the circle of radius 2, lying in the xy-plane centered at the origin. The parameterization of that circle is $\mathbf{c}(t) = (2\cos t, 2\sin t, 0)$, and thus $\mathbf{c}'(t) = (-2\sin t, 2\cos t, 0)$. It follows that $\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} (2(2\sin t), -2\cos t, 0) \cdot (-2\sin t, 2\cos t, 0)\, dt = \int_0^{2\pi} -8 + 4\cos^2 t\, dt = -12\pi$.

23. Write a *short* essay on each of the following topics: (i) Maxwell's Equations; (ii) conservative vector fields; (iii) local and global extrema.

Solution 23. Try and write concisely and coherently, using only as many symbols and equations as strictly necessary.
(i) Maxwell's equations: Maxwell's equations are fundamental to the modern theory of electromagnetism, studied intensely during the 19th century. You should at the very least state and label each of the equations, and give a one-sentence interpretation of each one.
(ii) Conservative vector fields are ones which are path independent, which turns out to be equivalent to being the gradient field of a scalar potential. One direction of the equivalence is not too hard to see, and you could prove it. Examples of conservative vector fields are the gravitational field of the earth, or a stationary electric point charge. Conservative vector fields give rise to exact differentials, a connection on which you could elaborate.
(iii) The critical points of a function $z = f(x, y)$ are determined by setting the x and y derivative equal to zero (why?). The discriminant informs us of the nature of the critical point (saddle, local min, local max). The global extrema of a function occur either at critical points or on the boundary of the given domain. Compare with the 1-dimensional case. Give an example.

Please send any comments or corrections to julia.wolf@cantab.net.