

**MITES 2010 ADVANCED CALCULUS  
REVIEW SHEET FOR MIDTERM-SKETCH SOLUTIONS**

1. Find the area enclosed by the curve  $r = 5 \sin \theta$  between  $\theta = 0$  and  $\theta = \pi/3$ .

**Solution 1.** The area is calculated as  $A = \frac{1}{2} \int_0^{\pi/3} f(\theta)^2 d\theta$ . In order to find the antiderivative of  $\sin^2 \theta$ , use the double angle formula (see formula sheet). The final answer is  $A = \frac{25}{4}(\frac{\pi}{3} - \frac{\sqrt{3}}{4})$ .

2. Find the length of the polar curve  $r = 4 \sin^2 \frac{\theta}{2}$  between  $\theta = 0$  and  $\theta = \pi$ .

**Solution 2.** The length of the polar curve is calculated as  $L = \int_0^\pi \sqrt{f(\theta)^2 + f'(\theta)^2} d\theta$ . We find that  $f(\theta)^2 + f'(\theta)^2 = (\sin^2(\theta/2))^2 + (\sin(\theta/2) \cos(\theta/2))^2 = \sin^2(\theta/2)$ , from which we obtain the answer  $L = 2$ .

3. Prove that (i)  $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})$ ; (ii)  $\mathbf{B} \cdot (\mathbf{A} \times \mathbf{C}) = -\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ ; (iii)  $\mathbf{C} \cdot (\mathbf{A} \times \mathbf{C}) = 0$ .

**Solution 3.** You can argue coordinate-wise or geometrically, remembering that the scalar triple product equals the volume of the parallelepiped spanned by the three vectors. (For those of you familiar with  $3 \times 3$  matrices, the scalar triple product  $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$  equals the determinant of the matrix with columns  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{C}$ .)

4. Show that  $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} \times \mathbf{c}) = (\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}))\mathbf{a}$ .

**Solution 4.** You can compute this quadruple vector product coordinate by coordinate, which is rather tedious. As a short-cut, you can use the identity from Question 9 on the first p-set, noting that the scalar triple product is zero when two of the vectors involved are identical.

5. Use the cross product to find the area of the triangle spanned by the three points  $(2, 1, 0)$ ,  $(3, 1, 5)$  and  $(0, 2, -1)$ .

**Solution 5.** We saw on the first p-set that the area of a triangle spanned by two vectors  $\mathbf{a}$  and  $\mathbf{b}$  originating at some point  $P$  is equal to  $\frac{1}{2}\|\mathbf{a} \times \mathbf{b}\|$ . Here we can take  $P = (0, 2, -1)$ ,  $\mathbf{a} = (3, 1, 5) - (0, 2, -1) = (3, -1, 6)$  and  $\mathbf{b} = (2, 1, 0) - (0, 2, -1) = (2, -1, 1)$ . We have  $\mathbf{a} \times \mathbf{b} = (5, 9, -1)$  and hence  $\frac{1}{2}\|\mathbf{a} \times \mathbf{b}\| = \frac{1}{2}\sqrt{107}$ .

6. If  $V = x^2 + y^2 + z^2$ , express  $x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} + z \frac{\partial V}{\partial z}$  in its simplest form.

**Solution 6.** After computing the appropriate partial derivatives, you should find that  $x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} + z \frac{\partial V}{\partial z} = 2(x^2 + y^2 + z^2) = 2V$ .

7. If  $V = 250$  volts and  $R = 50$  ohms, find the approximate change in  $I$  resulting from an increase of 1 volt in  $V$  and an increase of 0.5 ohm in  $R$ .

**Solution 7.** This problem is about small changes. From physics we know that  $I = V/R$ , and hence  $\Delta I \approx \frac{\partial I}{\partial V} \Delta V + \frac{\partial I}{\partial R} \Delta R$ . Clearly  $\frac{\partial I}{\partial V} = \frac{1}{R}$  and  $\frac{\partial I}{\partial R} = -\frac{V}{R^2}$ , and  $\Delta V = 1$  and  $\Delta R = 0.5$ . The result of the numerical computation is  $-0.03$  (ampere).

8. If  $z = 2xy - 3x^2y$  and  $x$  is increasing at  $2m/s$ , determine the rate at which  $y$  must be changing so that at the instant when  $x = 3m$  and  $y = 1m$ ,  $z$  is neither increasing nor decreasing.

**Solution 8.** This problem is about the chain rule. We want the rate at which  $z$  is changing,  $\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$ , to be equal to 0. We compute  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$  at  $x = 3$  and  $y = 1$ , set  $\frac{dx}{dt} = 2$  and solve for  $\frac{dy}{dt}$ . The answer is  $-32/21$ .

9. In what direction should one travel from the origin to obtain the most rapid decrease of the function  $f(x, y, z) = (2 - x - y)^2 + (3x + 2y - z + 1)^2$ ?

**Solution 9.** The direction of the most rapid increase is given by  $\nabla f$ , so the direction of the most rapid decrease is given by  $-\nabla f$  evaluated at the origin. The computation yields  $\nabla f_{(0,0,0)} = (2, 0, -2)$ .

10. Use the chain rule to evaluate  $\frac{\partial f}{\partial s}$  for  $f(x, y, z) = x^3y^2 - 2z$  with  $x = 2s^2t$ ,  $y = st^3$  and  $z = st^2$ .

**Solution 10.** More chain rule:  $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial s}$ . Applying this formula faithfully we obtain  $\frac{\partial f}{\partial s} = 32s^7t^9 - 2t^2$ .

11. Find  $\frac{\partial z}{\partial x}$  when  $x^2y + y^2z + xz^2 = 3$ .

**Solution 11.** By implicit differentiation, we have  $0 = \frac{\partial F}{\partial x} \frac{\partial x}{\partial x} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x}$ , where  $F = x^2y + y^2z + xz^2 - 3$ . Solving for  $\frac{\partial z}{\partial x}$ , we obtain  $\frac{\partial z}{\partial x} = -\frac{z^2 + 2xy}{y^2 + 2xz}$ .

12. If  $z = e^x(x \cos y - y \sin y)$ , show that  $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ .

**Solution 12.** This is just a computation involving the product rule.

**13.** Find the directional derivative of  $f(x, y, z) = 4xy + z^3$  at  $(3, -1, 1)$  in the direction pointing to the origin.

**Solution 13.** Computing the gradient, we find  $\nabla f = (4y, 4x, 3z^2)$ . The direction in question is  $\mathbf{u} = (-3, 1, -1)$ , which gives a unit vector  $\hat{\mathbf{u}} = \frac{1}{\sqrt{11}}(-3, 1, -1)$ . Therefore the directional derivative is  $D_{\hat{\mathbf{u}}}f(P) = \nabla f_P \cdot \hat{\mathbf{u}} = (-4, 12, 3) \cdot \frac{1}{\sqrt{11}}(-3, 1, -1) = \frac{21}{\sqrt{11}}$ .

**14.** Find a function  $f$  such that  $\nabla f = (z, 2y, x)$ .

**Solution 14.** We have  $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) = (z, 2y, x)$ , so that  $f(x, y, z) = zx + g_1(y, z)$ ,  $f(x, y, z) = y^2 + g_2(x, z)$  and  $f(x, y, z) = xz + g_3(x, y)$  for some two-variable functions  $g_1, g_2$  and  $g_3$ . A function  $f$  that satisfies all three equations is  $f(x, y, z) = xz + y^2$  (plus any constant).

**15.** Find an equation of the tangent plane to the surface  $x^2 + 3y^2 + 4z^2 = 20$  at  $(2, 2, 1)$ .

**Solution 15.** In order to find the normal to the surface we compute  $\nabla F = (2x, 6y, 8z)$ . Evaluated at  $(2, 2, 1)$  this gives the normal vector  $(4, 12, 8)$ , and thus the equation of the tangent plane is  $(\mathbf{x} - (2, 2, 1)) \cdot (4, 12, 8) = 0$ .

**16.** Determine the critical points of the function  $f(x, y) = (x^2 + y^2)e^{-x}$ , and use the Second Derivative Test to establish whether they are local maxima, minima or saddle points.

**Solution 16.** The critical points of  $f$  satisfy  $\frac{\partial f}{\partial x} = (2x - (x^2 + y^2))e^{-x} = 0$  and  $\frac{\partial f}{\partial y} = 2ye^{-x} = 0$ , so  $y = 0$  from the latter equation and  $x = 2, x = 0$  from the former. So the two critical points are  $(0, 0)$  and  $(2, 0)$ . We compute the discriminant  $D = \frac{\partial^2 f}{\partial^2 x} \frac{\partial^2 f}{\partial^2 y} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$  at both points, and find that  $(0, 0)$  is a local minimum ( $D > 0$  and  $\frac{\partial^2 f}{\partial^2 x} > 0$ ) and  $(2, 0)$  is a saddle ( $D < 0$ ).

**17.** Find the global extrema of the function  $f(x, y) = x^3 + x^2y + 2y^2$  on the domain  $x, y \geq 0$ ,  $x + y \leq 1$ .

**Solution 17.** The critical points of  $f$  satisfy  $\frac{\partial f}{\partial x} = 3x^2 + 2xy = 0$  and  $\frac{\partial f}{\partial y} = x^2 + 4y = 0$ . Plugging the latter into the former, we get  $-12y + 2xy = 0$ , which means either  $y = 0$  (and thus  $x = 0$ ) or  $x = 6$  (and  $y = -9$ ). The second point lies outside the domain of interest, so we are not going to consider it further. The first point has zero discriminant, so the second derivative test

is inconclusive, but it lies on the boundary (with  $f(0,0) = 0$ ), which we will need to examine anyway: When  $x = 0$ ,  $f(0,y) = 2y^2$ , which has a min of 0 and a max of 2 for  $0 \leq y \leq 1$ . When  $y = 0$ ,  $f(x,0) = x^3$ , which has a min of 0 and a max of 1 for  $0 \leq x \leq 1$ . Finally, when  $x + y = 1$ ,  $f(x,1-x) = 2 - x^2$ , which has a min of 1 and a max of 2 for  $0 \leq x \leq 1$ . Therefore we have a global maximum at  $(0,1)$  (with  $f(0,1) = 2$ ) and a global min at  $(0,0)$  (with  $f(0,0) = 0$ ).

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