Mini-Revision Sheet: Solutions

1. What does it mean for a set $A$ to be countable? Is the set of all rationals with prime denominator countable? Justify your answer.

A set $A$ is said to be countable if and only if there is a bijection $f : A \rightarrow \mathbb{N}$. The set of all rationals with prime denominator is countable because it is an infinite subset of a countable set.

2. What does it mean for the cardinality of a set $X$ to be greater than or equal to the cardinality of a set $Y$? What is the relationship between the cardinality of $\mathbb{N}$ and the cardinality of the set of functions $f : \mathbb{N} \rightarrow \{0, 1\}$?

The cardinality of $X$ is said to be greater than or equal to the cardinality of a set $Y$ if there exists an injection $f : Y \rightarrow X$. The set of functions $f : \mathbb{N} \rightarrow \{0, 1\}$ can be identified with the power set of $\mathbb{N}$ (using, for each subset $A \subseteq \mathbb{N}$, the indicator function $1_A : \mathbb{N} \rightarrow \{0, 1\}$, defined by $1_A(x) = 1$ if $x \in A$, and $1_A(x) = 0$ if $x \notin A$). But we showed that $\mathbb{N} \prec 2^\mathbb{N}$, i.e. that the cardinality of $2^\mathbb{N}$ is greater than but not equal to the cardinality of $\mathbb{N}$.

3. Exhibit a bijection that shows that $(0, 1]$ is a continuum.

A suitable bijection is the function $f : (0, 1] \rightarrow (0, 1)$, defined by $f(x) = 1/(n + 1)$ if $x = 1/n$ for some $n \in \mathbb{N}$, and $f(x) = x$ otherwise.

4. True or false? If $(a_n)_{n \in \mathbb{N}}$ has a bounded subsequence, then $(a_n)_{n \in \mathbb{N}}$ has a convergent subsequence. Justify your answer.

True. Let $(a_m(k))_{k \in \mathbb{N}}$ be a bounded subsequence of $(a_n)_{n \in \mathbb{N}}$. Then by the Bolzano-Weierstrass theorem, $(a_m(k))_{k \in \mathbb{N}}$ contains a convergent subsequence. But a subsequence of $(a_m(k))_{k \in \mathbb{N}}$ is a subsequence of the original sequence $(a_n)_{n \in \mathbb{N}}$.

5. Give two equivalent definitions of what it means for an element $a \in \mathbb{R}$ to be an accumulation point of the sequence $(a_n)_{n \in \mathbb{N}}$.

We say $a \in \mathbb{R}$ is an accumulation point of the sequence $(a_n)_{n \in \mathbb{N}}$ if there is a subsequence of $(a_n)_{n \in \mathbb{N}}$ which converges to $a$. Equivalently, $a \in \mathbb{R}$ is an accumulation point of $(a_n)_{n \in \mathbb{N}}$ if and only if for all $\varepsilon > 0$, the set

$$\{n \in \mathbb{N} : a - \varepsilon < a_n < a + \varepsilon\}$$

is infinite.

6. What is the $\liminf_{n \to \infty}$ of the sequence $(a_n)_{n \in \mathbb{N}}$ defined by $a_{2n} = 2n, a_{2n+1} = 1 + \frac{1}{n}$ for all $n \in \mathbb{N}$?

This sequence is bounded below (by 1) but not bounded above. However, the $\liminf_{n \to \infty}$ of this sequence is still well-defined and equals 1. You can see this either by using
the formula \( \liminf_{n \to \infty} a_n = \lim_{n \to \infty} \inf_{k \geq n} a_k \) we derived, or by analyzing the set of accumulation points, which consists of 1 (being the limit of the subsequence consisting of the odd terms only) and \( \infty \) (being the limit of the subsequence consisting of the even terms) and therefore has infimum 1.

7. Let \((a_n)_{n \in \mathbb{N}}\) be defined by \(a_n = (-1)^{n^2}\) for all \(n \in \mathbb{N}\). Show that \((a_n)_{n \in \mathbb{N}}\) is not a Cauchy sequence, and deduce that \((a_n)_{n \in \mathbb{N}}\) does not converge.

Observe that \(a_n\) is equal to \(-1\) if and only if \(n\) is odd, and equal to 1 otherwise. Let \(\varepsilon = 1\). Then for all \(N \in \mathbb{N}\), there exist \(p, q \geq N\) (pick \(p = N, q = N + 1\)) such that |\(a_p - a_q\)| = |\(1 - (-1)\)| = 2 \(\geq 1\). So \((a_n)_{n \in \mathbb{N}}\) is not a Cauchy sequence. By Cauchy’s theorem (the general principle of convergence), this means that the sequence \((a_n)_{n \in \mathbb{N}}\) does not converge.

8. Define what it means for a function to be uniformly continuous on its domain, and state Cantor’s theorem on uniform continuity.

Let \(A \subseteq \mathbb{R}\) and let \(f : A \to \mathbb{R}\) be a function. Then \(f\) is uniformly continuous on \(A\) if

\[
(\forall \varepsilon > 0) \ (\exists \delta > 0) \ (\forall x, y \in A) \ [(|x - y| < \delta) \implies |f(x) - f(a)| < \varepsilon].
\]

Cantor’s theorem on uniform continuity states that any continuous function on a closed bounded interval is uniformly convergent.

9. Is the sequence of functions \(f_n : [0, \pi] \to \mathbb{R}\) defined by \(f_n(x) = \sin^n(x)\) uniformly convergent? Justify your answer.

No. The pointwise limit of the sequence \((f_n)_{n \in \mathbb{N}}\) is the function \(f : [0, \pi] \to \mathbb{R}\) which is defined by \(f(x) = 0\) when \(x \in [0, \pi] \setminus \{\pi/2\}\) and \(f(x) = 1\) at \(x = \pi/2\). This is a discontinuous function, so by Weierstrass’s theorem on uniform convergence the convergence cannot be uniform.