Further Topics in Analysis: Supplementary Exercises

1. Let \( F = \{ f : (0, 1) \to \mathbb{R} \} \) be the set of all functions from the segment \((0, 1)\) to \(\mathbb{R}\). Prove that \( \text{card}((0, 1)) < \text{card}(F) \).

   **Hint:** Consider the set of characteristic functions
   \[
   f_A(x) = \begin{cases} 
   1, & \text{if } x \in A, \\
   0, & \text{if } x \notin A,
   \end{cases}
   \]
   where \( A \subset (0, 1) \) is a subset of \((0, 1)\).

2. Let \( L \) be the set of all straight lines \( l \) on the plane \( \mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \). Prove that \( L \) is a continuum.

3. A real number \( x \) is called **algebraic** if \( x \) is a solution of a non-zero polynomial equation
   \[
   (\ast) \quad a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0
   \]
   with integer coefficients \( a_n, \ldots, a_0 \in \mathbb{Z} \).
   
   For \( m \in \mathbb{N} \), let \( A_m \) be the set of algebraic numbers that are solutions of equations such as \((\ast)\) where
   \[
   n + |a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0| \leq m.
   \]
   (i) Show that, for each \( m \in \mathbb{N} \), \( A_m \) is a finite set.
   (ii) Deduce that the set of algebraic numbers is countable.
   (iii) Deduce that there are real numbers that are not algebraic.

4. Let \((a_n)_{n \in \mathbb{N}}\) be a sequence. Prove the following two statements.
   (a) If \( a_n \to a \) as \( n \to \infty \) then any subsequence \( a_{m(k)} \to a \) as \( k \to \infty \).
   (b) If \((a_n)_{n \in \mathbb{N}}\) is bounded then any subsequence \((a_{m(k)})_{k \in \mathbb{N}}\) is bounded.

5. Describe the set of all accumulation points of the following sequences.
   (a) \( a_n = \sqrt{n+1} \);
   (b) \( a_n = \sqrt{n+1} + (-1)^n\sqrt{n-1} \);
   (c) \( a_n = \frac{\sqrt{n+1} + (-1)^n\sqrt{n-1}}{n} \);
   (d) \( a_n = \frac{\sqrt{n+1} + (-1)^n\sqrt{n-1}}{\sqrt{n}} \).
6. Let \((a_n)_{n \in \mathbb{N}}\) be a sequence in \([0, \infty)\). Let us agree the following. If \((a_n)_{n \in \mathbb{N}}\) is not bounded above, then

\[
\limsup_{n \to \infty} a_n = +\infty.
\]

Show that

\[
\limsup_{n \to \infty} a_n
\]

may be defined for any sequence \((a_n)_{n \in \mathbb{N}}\) in \((0, \infty)\), and takes values in

\([0, \infty) \cup \{+\infty\}\).

7. Let \((a_n)_{n \in \mathbb{N}^+}\) be a bounded sequence in \(\mathbb{R}\).

(a) Define what it means to say that \(a \in \mathbb{R}\) is an accumulation point of \((a_n)_{n \in \mathbb{N}^+}\).

(b) Give a characterisation of the property that \(a \in \mathbb{R}\) is an accumulation point in terms of intervals of the form \((a - \varepsilon, a + \varepsilon)\) \((\varepsilon > 0)\).

(c) Define what it means to say that \(\alpha = \limsup_{n \to \infty} a_n\) \((\alpha \in \mathbb{R})\).

(d) Give a characterisation of the property that \(\alpha = \limsup_{n \to \infty} a_n\) in terms of intervals of the form \((\alpha - \varepsilon, +\infty)\) and \((\alpha + \varepsilon, +\infty)\) \((\varepsilon > 0)\).

(e) Define what it means to say that \(\beta = \liminf_{n \to \infty} a_n\) \((\beta \in \mathbb{R})\).

(f) Give a characterisation of the property that \(\beta = \liminf_{n \to \infty} a_n\) in terms of intervals of the form \((-\infty, \beta - \varepsilon)\) and \((-\infty, \beta + \varepsilon)\) \((\varepsilon > 0)\).

8. (a) Give an example of bounded sequences \((a_n)_{n \in \mathbb{N}}\) and \((b_n)_{n \in \mathbb{N}}\) of real numbers where

\[
\limsup_{n \to \infty} (a_n + b_n) \neq \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n.
\]

(b) By considering \(\sup\{a_m + b_m : m \geq n\}\) prove that

\[
\limsup_{n \to \infty} (a_n + b_n) \leq \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n.
\]

(c) What is the corresponding result for \(\liminf_{n \to \infty}(a_n + b_n)\)?

9. Let \((a_n)_{n \in \mathbb{N}^+}\) be a bounded sequence in \(\mathbb{R}\). For \(n \in \mathbb{N}^+\), set

\[
b_n := \sup_{k \geq n} a_k,
\]

\[
c_n := \inf_{k \geq n} a_k.
\]

(a) Explain why \(\alpha := \lim_{n \to \infty} b_n\) exists in \(\mathbb{R}\).

(b) Explain why \(\beta := \lim_{n \to \infty} c_n\) exists in \(\mathbb{R}\).

10. Using the definition, verify directly that the following sequences \((a_n)_{n \in \mathbb{N}}\) are Cauchy sequences.
(a) \( a_n = \frac{n-1}{n+1} \)
(b) \( a_n = \frac{\sqrt{n^2+1}}{n} \)

11. For each of the following functions \( f : [0, 1] \to \mathbb{R} \), prove by using the definition that the function is uniformly continuous on the segment \([0, 1]\).

(a) \( f(x) = 1 \)
(b) \( f(x) = x \)
(c) \( f(x) = x^2 \)

12. Let \( a > 0 \). Define
\[
f : \mathbb{R} \to \mathbb{R}; x \to \frac{1}{a + x^2}.
\]
Show that \( f \) is uniformly continuous on \( \mathbb{R} \).

13. (a) Show directly from the definition that the function \( f : [0, \infty) \to \mathbb{R} \) defined by \( f(x) = \sqrt{x} \) is uniformly continuous on \([0, \infty]\).
(b) Show that the function \( g : \mathbb{R} \to \mathbb{R} \) defined by \( g(x) = x^3 \) is not uniformly continuous on \( \mathbb{R} \).

14. Let \( A \) be a subset of the real line and \((f_n(x))_{n \in \mathbb{N}}\) be a sequence of functions from \( A \) to \( \mathbb{R} \).

(a) What does it mean to say that the sequence \((f_n(x))_{n \in \mathbb{N}}\) converges uniformly on \( A \) to a function \( f(x) \)?
(b) Prove that the sequence of functions \( f_n : [0, 1] \to \mathbb{R} \), given, for all \( n \in \mathbb{N} \), by
\[
f_n(x) = \frac{nx^2 + 2x}{nx + 1},
\]
converges uniformly on the closed interval \([0, 1]\) to the function \( f(x) = x \).

15. Let \( A \) be a subset of the real line and \((f_n(x))_{n \in \mathbb{N}}\) be a sequence of functions from \( A \) to \( \mathbb{R} \).

(a) State and prove Weierstrass’s Theorem on Uniform Convergence.
(b) Prove that for any \( a \in (0, 1) \), the sequence of functions \( f_n : [0, a] \to \mathbb{R} \), given, for all \( n \in \mathbb{N} \), by
\[
f_n(x) = \frac{x^n}{1 + x^{n+1}},
\]
converges uniformly on the closed interval \([0, a]\), but it does not converge uniformly on \([0, 1]\).

16. Give an example of a sequence of functions \((f_n)_{n \in \mathbb{N}}\) together with a function \( f \) satisfying the following properties.
(i) For each $n \in \mathbb{N}$, the function $f_n : [0, 1] \to \mathbb{R}$ is continuous.
(ii) The function $f : [0, 1] \to \mathbb{R}$ is continuous.
(iii) The sequence $(f_n)_{n \in \mathbb{N}}$ converges to $f$ pointwise but not uniformly.

17. Let $f : (0, \infty) \to \mathbb{R}$ be the function defined by $f(x) = \log x$.

(a) By considering the partitions
$$Q_n = \{1, e^{1/n}, e^{2/n}, \ldots, e\},$$
show that $f$ is Riemann integrable over $[1, e]$.

(b) Using the formula
$$\lim_{n \to \infty} n(e^{1/n} - 1) = 1$$
find the value of the integral in Part (a).

18. Show that the function $g : [0, 1] \to \mathbb{R}$ defined by
$$g(x) = \begin{cases} \sin(1/x) & \text{if } x \in (0, 1] \\ 0 & \text{if } x = 0 \end{cases}$$
is Riemann integrable on $[0, 1]$.

19. Let $h : [0, 1] \to \mathbb{R}$ be a function with the following properties.

(i) For all $x \in [0, 1]$, $0 \leq h(x) \leq 1$.

(ii) For all $\varepsilon > 0$, the set
$$\{x \in [0, 1] : h(x) \geq \varepsilon\}$$
has finite cardinality, bounded by some constant $K(\varepsilon)$ which depends only on $\varepsilon$.

Prove that $h$ is Riemann integrable on $[0, 1]$, and evaluate
$$\int_0^1 h(x) \, dx.$$